New quantum invariants and knot fiberedness

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joint work with P. Orland, T. Saunders-A'Court, J. Svoboda
arXiv:2512.23700

Stanford Topology Seminar

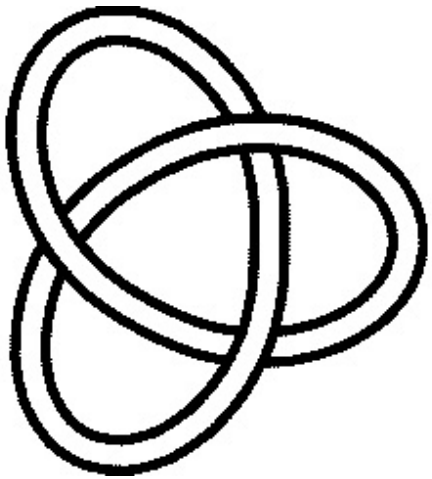
April 7, 2026

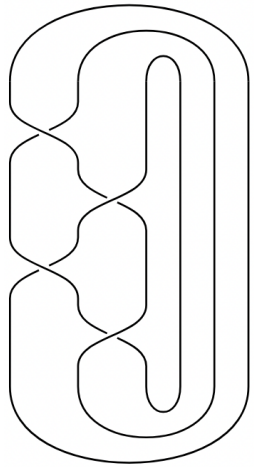
Motivation

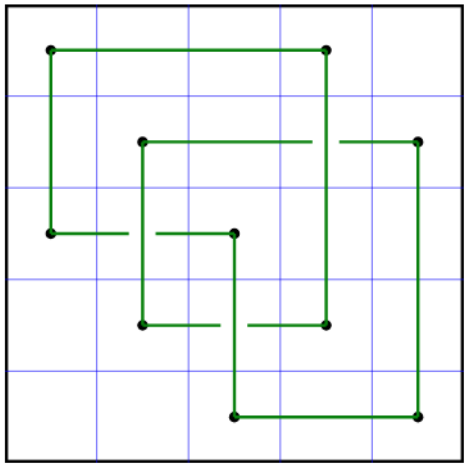
Quantum invariants of knots

Let $K \subset S^3$ be a knot (embedding of a circle in the 3-sphere). Represent K in the plane by a knot diagram D.

Commonly used knot diagrams:





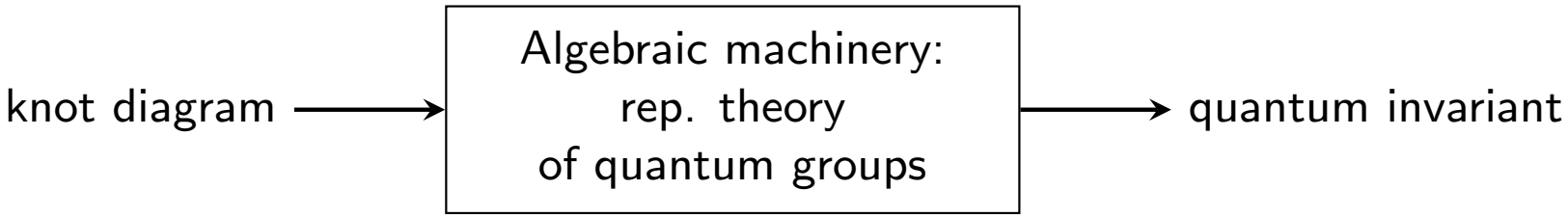


Key in the study of low-dimensional topology.

Motivation

Quantum invariants of knots

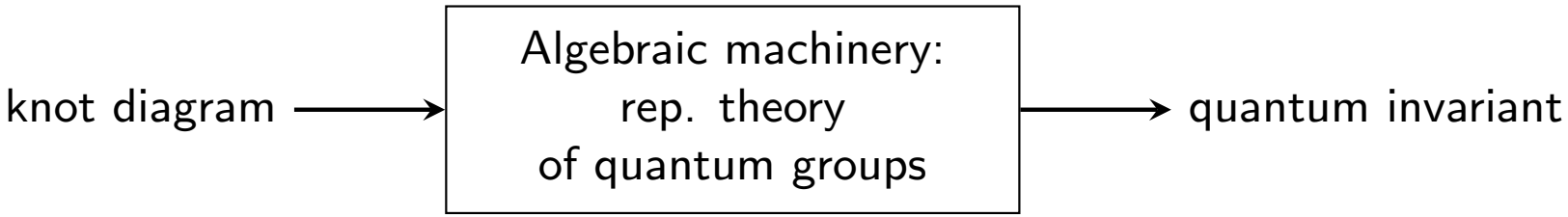
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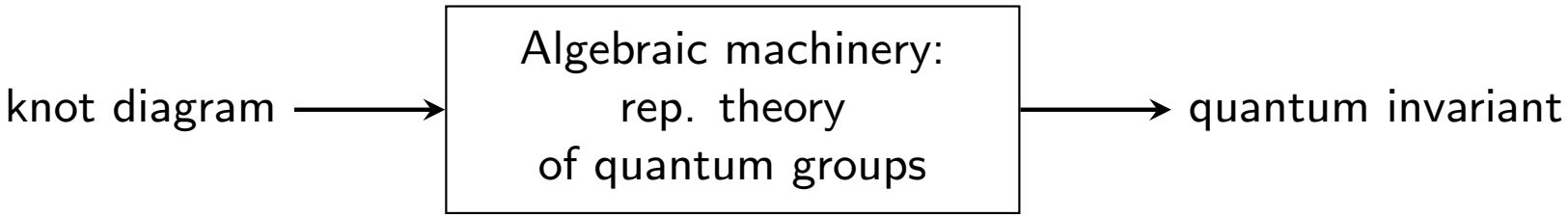
Why should we care about quantum topology?:

1. Strong distinguishing power.
2. Detects 4-dimensional information, incl. smooth vs. topological structures.

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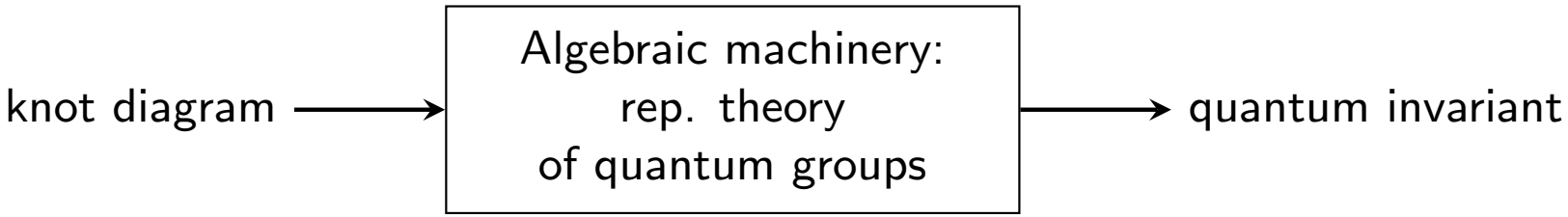
Question: what geometric content is read by quantum knot invariants?

Striking conjectures! $\rightsquigarrow$ volume, incompressible surfaces, genus...

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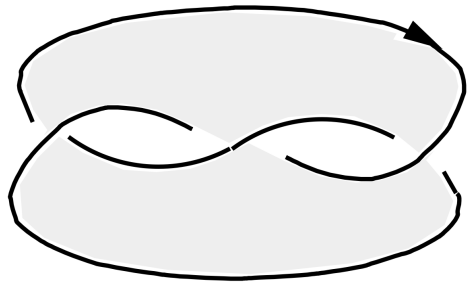
Striking conjectures! $\rightsquigarrow$ volume, incompressible surfaces, genus...

Our work: the $\mathfrak{sl}_2$ semisimple quantum knot invariants read existence of fibration over the knot complement.

Knots and fiberedness

Knots and surfaces

Knot $K \subset S^3 \rightsquigarrow$ connected, cpt, oriented surface S smoothly embedded in S^3, $\partial S = K$.
We call such a surface a Seifert surface of K.
Minimum genus of such surfaces $\rightsquigarrow g(K)$.

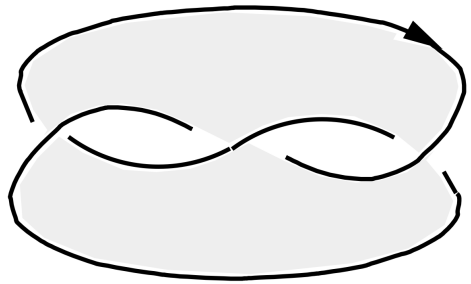


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Figure: N. Saveliev.

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What can we use Seifert surfaces for? We can construct the *Alexander polynomial* of a knot K.

Example: figure-eight

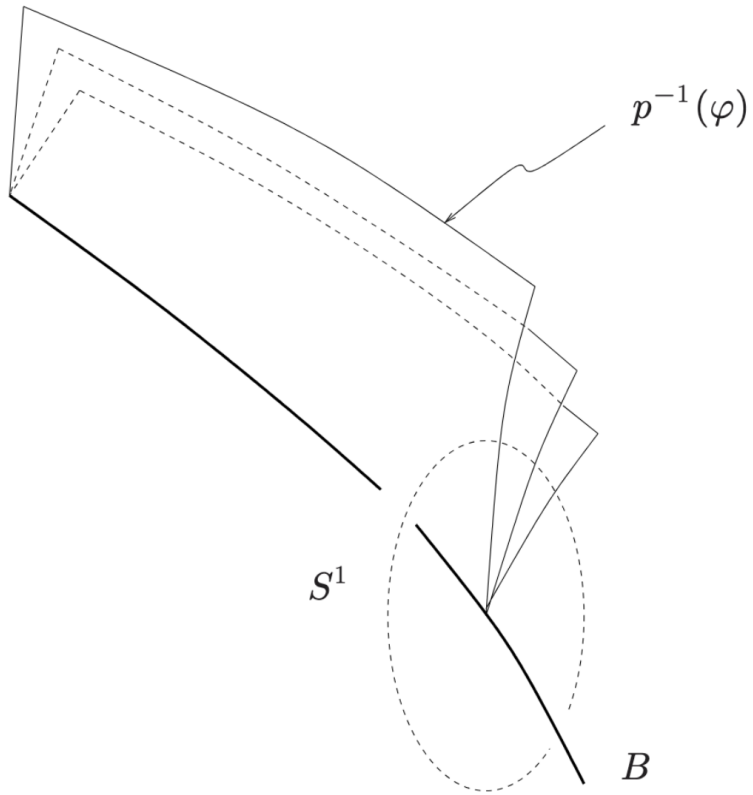
$$\Delta_{4_1}(x) = x^{-1} - 3 + x$$

Figure: N. Saveliev.

Fibered knots

Definition: A knot $K \subset S^3$ is fibered if there exists a fibration $S^3 \setminus K \to S^1$ with fiber F, $\partial F = K$.

We call the closure of F the (unique) fibered surface of K.



Equivalently, fibered knots are the binding of open book decompositions of S^3.

Figure: H. Geiges.

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Proposition: $g(F) = g(K)$.

The Alexander polynomial is the characteristic polynomial of the map induced by the monodromy $h : F \to F$ on the first homology.

Theorem: Alexander polynomial of a fibered knot is monic and has degree $g(K)$.

$$\Delta_{4_1}(x) = x^{-1} - 3 + x$$

Hopf invariant or enhanced Milnor number

Let $N(K) \simeq D^2 \times K$ be a neighbourhood of K. We define a vector field ξ_K in S^3:

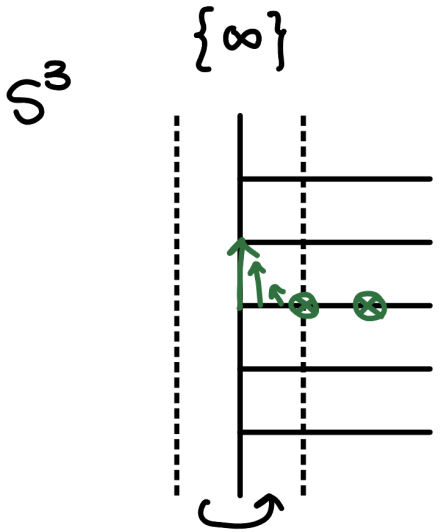
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Example: unknot.



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The enhanced Milnor number $\lambda(K)$ is the homotopy class of the plane field ξ_K.
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Equivalently, we can identify ξ_K with the contact structure induced by the open book decomposition of S^3, which is bijection with $\mathbb{Z} \cup \{*\}$ (given by the Hopf map h).
$\rightsquigarrow \lambda(K) = -h(\xi) \in \mathbb{Z}$.

Hopf invariant or enhanced Milnor number

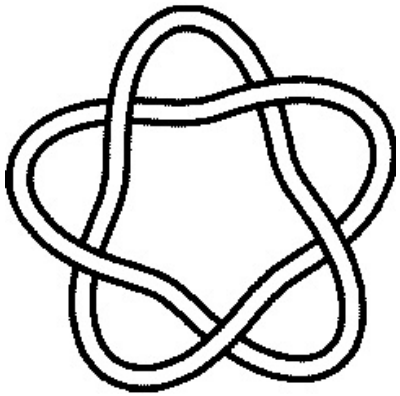
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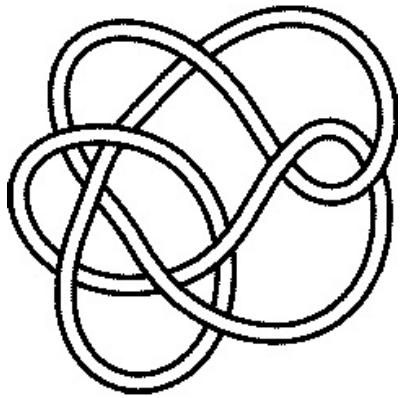
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Example: the knots 5_1 and 10_{132} have the same genus and Alexander polynomial, but different Hopf invariant!

$$\Delta_{5_1}(x) = \Delta_{10_{132}}(x) = x^{-2} - x^{-1} + 1 - x + x^2$$



$\lambda(5_1) = 0$



$\lambda(10_{132}) = 2$

Figures: katlas.org.

Knot Floer homology

Knot Floer homology (Ozváth–Szabó, Rasmussen):

$$\widehat{HFK}(K) = \bigoplus_{n,s\in\mathbb{Z}} \widehat{HFK}_s(K,n).$$

Theorem: The graded Euler characteristic of $\widehat{HFK}$ is the Alexander polynomial $\Delta_K(x)$.

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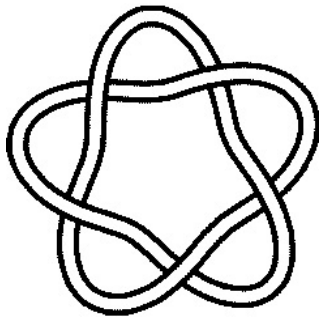
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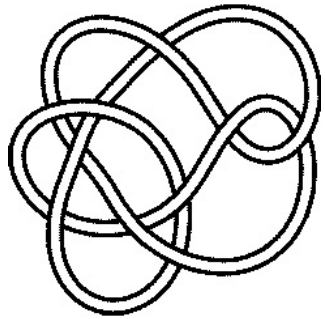
Examples:

$$\pi\left(\widehat{HFK}(5_1)\right) = m^{-4}x^{-2} + m^{-3}x^{-1} + m^{-2} + m^{-1}x + x^2$$

$$\pi\left(\widehat{HFK}(10_{132})\right) = m^{-2}x^{-2} + (2m^{-1}+1)x^{-1} + (2+m) + (2m+m^2)x + m^2x^2$$



5_1



10_{132}

Figures: katlas.org.

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Theorem: $\widehat{HFK}$ detects the genus: $\max\{n \mid \widehat{HFK}(K,n) \neq 0\} = g$.

Theorem (Ghiggini, Yi): $\widehat{HFK}$ detects fiberedness:

$$K \text{ is fibered} \iff \widehat{HFK}_s(K,g) = \begin{cases} \mathbb{Z} & s = \lambda(K) \quad \leftarrow \text{Hopf invariant} \\ 0 & \text{else} \end{cases}$$

We want to study whether quantum invariants detect fiberedness...

... But why do we care about quantum knot invariants in the first place?

Polynomials: Conjectured to be related to the knot volume, slopes of incompressible surfaces in the knot complement...

$$K \rightsquigarrow \{J_n(q)\}_{n\in\mathbb{Z}\geq 0} \quad \leftarrow \text{colored Jones polynomials}$$

(Jones, Witten, Reshetikhin–Turaev, Garoufalidis–Le)

Polynomials: Conjectured to be related to the knot volume, slopes of incompressible surfaces in the knot complement...

$$K \rightsquigarrow \{J_n(q)\}_{n\in\mathbb{Z}\geq 0}$$

Power series: The limit $q \nearrow 1$ of the sequence $\{J_n(q)\}_{n\in\mathbb{Z}\geq 0}$ agrees with

$$\frac{1}{\Delta_K(x)} + (q-1)\frac{P_K^{(1)}(x)}{\Delta_K^3(x)} + \cdots \in \mathbb{Q}[[x, q-1]]$$

where $x = q^n$. (Melvin–Morton, Rozansky, Bar Natan–Garoufalidis)

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It can be resumed as a 2-variable power series

$$F_K(x,q) = \sum_{i\in\frac{1}{2}+\mathbb{Z}\geq 0} f_i(q)x^i$$

for a certain class of knots, conjecturally all (Gukov–Manolescu, Park).

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Homology theories: $J_n(q)$ admits a categorification (Khovanov, Khovanov–Rozansky).

$F_K(x,q)$ is conjectured to admit a categorification as well, which would answer one of the central questions in quantum topology (Gukov–Pei–Putrov–Vafa).

Quantum knot invariants associated with $U_q(\mathfrak{sl}_2)$

State sum model

The quantum group $U_q(\mathfrak{sl}_2)$ is a "q-deformation" of $\mathfrak{sl}_2$. Generated by $\{E, F, K, H\}$ and the set of relations:

$$[H, E] = 2E, \quad [H, F] = 2F, \quad [E, F] = \frac{K - K^{-1}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}, \quad K^{\pm\frac{1}{2}} = q^{\pm\frac{H}{4}}$$

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and braiding

$$R = q^{\frac{H\otimes H}{4}} \sum_{k\geq 0} \frac{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^k}{[k]_q} q^{k(k-1)/4} E^k \otimes F^k \in U_q(\mathfrak{sl}_2)\hat{\otimes}U_q(\mathfrak{sl}_2)$$

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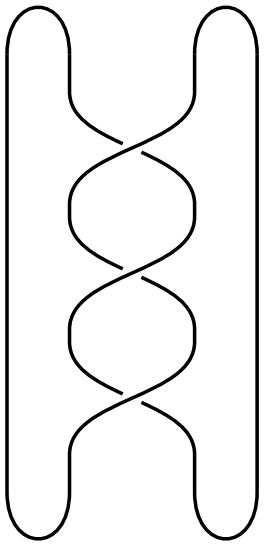
where $[k]_q := \frac{q^{\frac{k}{2}} - q^{-\frac{k}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$.

The action of R on the tensor product of $U_q(\mathfrak{sl}_2)$ representations induces a representation of the braid group.

Example of the state sum: trefoil

For generic q: {simple f.d. representation V_n} $\leftrightarrow$ {dim. of the vector space $(n+1)$}.

Example: trefoil. The semisimple invariants (colored Jones) are computed via:



$$\xrightarrow{J_n} \operatorname{tr}^q_{V_n \otimes V_n} \begin{pmatrix} V_n \otimes V_n \\ \uparrow R \\ V_n \otimes V_n \\ \uparrow R \\ V_n \otimes V_n \\ \uparrow R \\ V_n \otimes V_n \end{pmatrix} = \begin{cases} q + q^3 - q^4 & n = 1 \\ q^2 + q^5 - q^7 + q^8 - q^9 - q^{10} + q^{11} & n = 2 \\ \dots \end{cases}$$

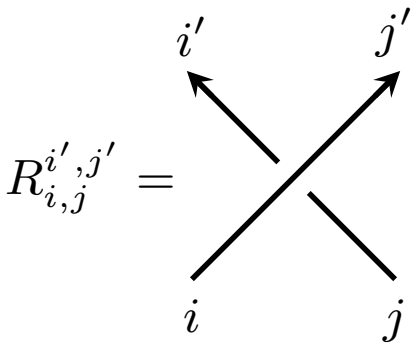
Initial construction of the F_K series

Let $V^h_{q^{-1}x}$ be the infinite-dimensional representation with highest weight $q^{-1}x$.

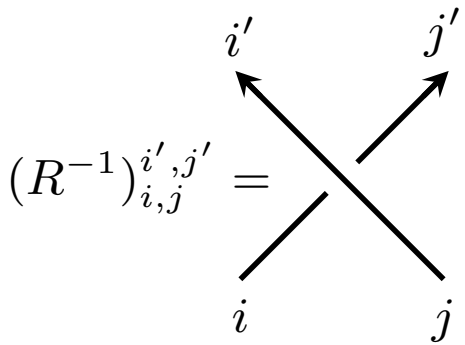
The R-matrix acting on $V^h_{q^{-1}x} \otimes V^h_{q^{-1}x}$ is:

$$R^{\pm 1} \cdot (v_i \otimes v_j) = \sum_{i+j=i'+j'} (R^{\pm 1})^{i',j'}_{i,j} (v_{i'} \otimes v_{j'})$$

In pictures:



Positive crossing



Negative crossing

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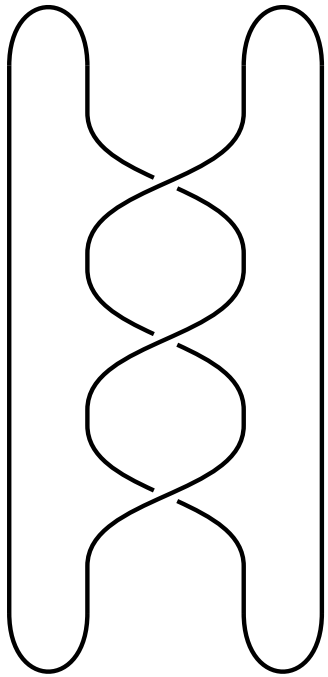
where

$$R^{i',j'}_{i,j} = \text{(polynomial in } q\text{)}\ x^{\frac{j+j'+1}{2}} (q^{j+1}x;q)_{i-j'} \in \mathbb{Z}[q][x]$$

and $\left(R^{-1}\right)^{i',j'}_{i,j} = R^{j',i'}_{j,i}\Big|_{\substack{q \to q^{-1} \\ x \to x^{-1}}} \in \mathbb{Z}[q^{-1}][x^{-1}]$.

$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1})$.

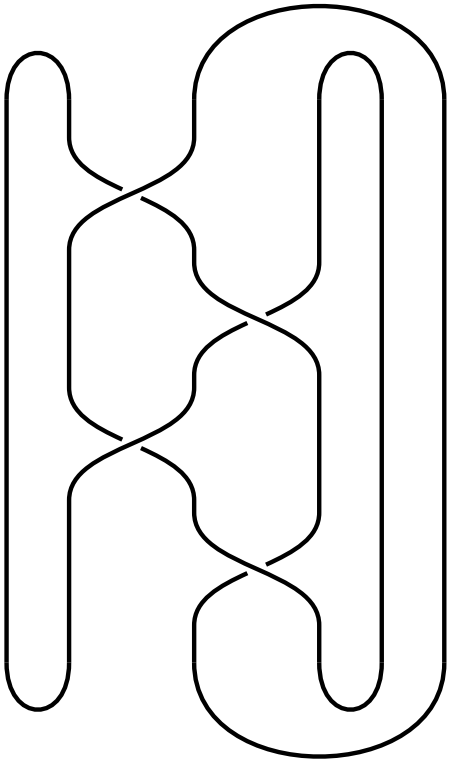
Trefoil knot



$$\xmapsto{F_K} \operatorname{tr}^q_{V\otimes V}\begin{pmatrix} V\otimes V \\ \uparrow R \\ V\otimes V \\ \uparrow R \\ V\otimes V \\ \uparrow R \\ V\otimes V \end{pmatrix} = \sum_{i\geq 0} R^{i,0}_{0,i} R^{i,0}_{i,0} R^{0,i}_{i,0}$$

$$= qx - q^2x^3 - q^3x^4 + q^6x^6 + q^8x^7 - q^{13}x^9 - q^{16}x^{10} + q^{23}x^{12} + q^{27}x^{13} - q^{36}x^{15} + \cdots$$

Figure eight knot



$$\xmapsto{F_K} \operatorname{tr}^q_{V^{\otimes 3}}(\beta) = \sum_{i,j\geq 0} (R^{-1})^{i,j}_{i,j}\, R^{i,0}_{0,i}\, (R^{-1})^{0,j}_{0,j}\, R^{0,i}_{i,0}$$

$$= \sum_{i,j\geq 0} g_{i,j}(x,q)\, x^{i-2j}, \text{ where } g_{i,j}(x,q) \in \mathbb{Z}[q,q^{-1},x].$$

Extension to fibered knots

Main issue: We need to modify the R-matrices to ensure the quantum trace gives us **only positive powers** of x.

$$R_{i,j}^{i',j'} = (\text{polynomial in } q)\; x^{\frac{j+j'+1}{2}} (q^{j+1}x;q)_{i-j'}$$

$$(R^{-1})_{i,j}^{i',j'} = (\text{polynomial in } q^{-1})\; x^{-\frac{i+i'+1}{2}} (q^{-i-1}x^{-1};q^{-1})_{j-i'}$$

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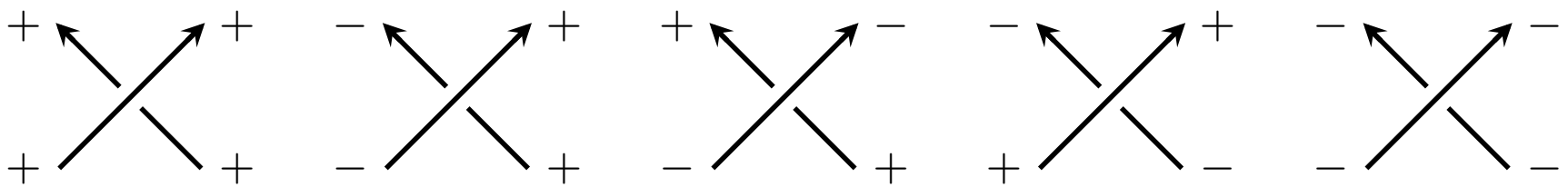
$$R_{i,j}^{i',j'} = O(x^{\frac{j+j'+1}{2}})$$
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$\rightsquigarrow$ It "makes sense" to extend the maps R and R^{-1} for certain regions of $\{(i,j,i',j')\} \in \mathbb{Z}^4$...



Extension to fibered knots

Main issue: We need to modify the R-matrices to ensure the quantum trace gives us **only positive powers** of x.

$$R_{i,j}^{i',j'} = O(x^{\frac{j+j'+1}{2}})$$
$$(R^{-1})_{i,j}^{i',j'} = O(x^{-\frac{j+j'+1}{2}})$$

$\rightsquigarrow$ It "makes sense" to extend the maps R and R^{-1} for certain regions of $\{(i,j,i',j')\} \in \mathbb{Z}^4$...

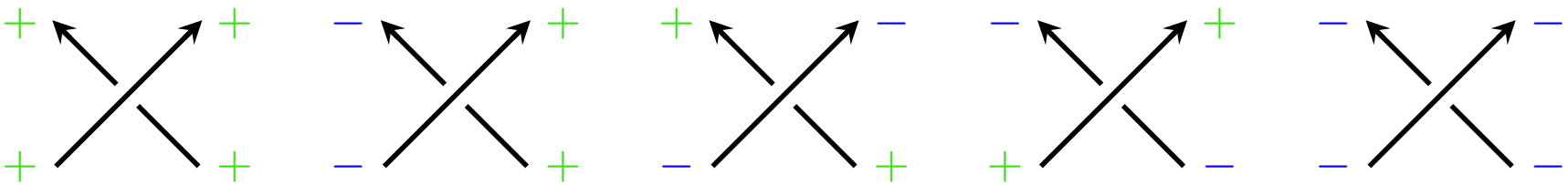
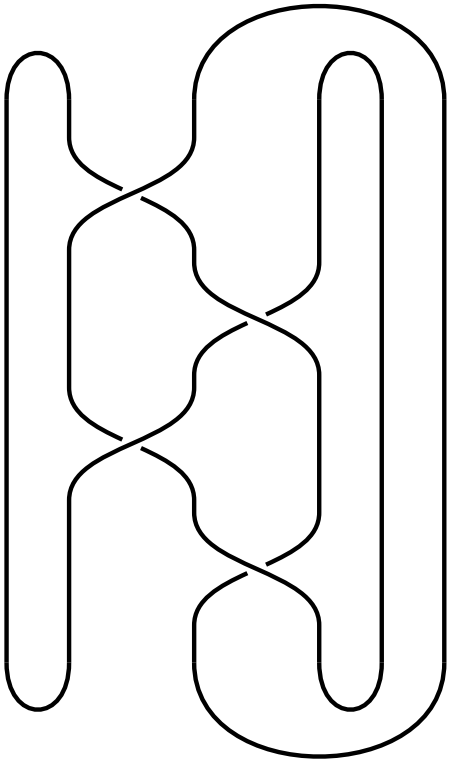


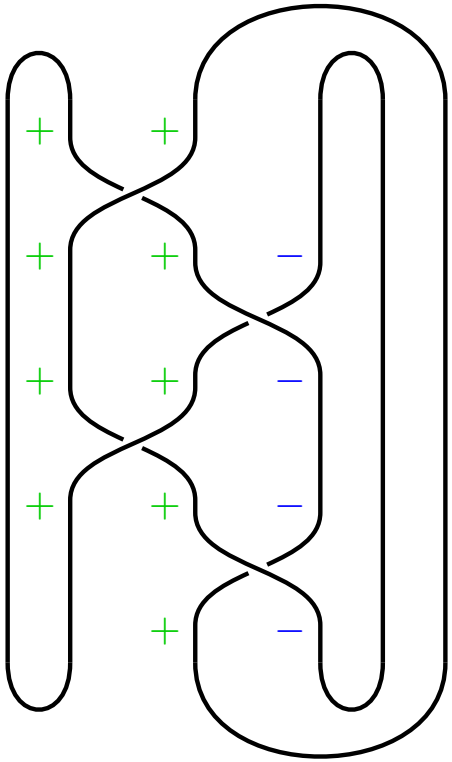
Figure eight knot



$$\xmapsto{F_K} \operatorname{tr}^q_{V^{\otimes 3}}(\beta) = \sum_{i,j\geq 0} (R^{-1})^{i,j}_{i,j}\, R^{i,0}_{0,i}\, (R^{-1})^{0,j}_{0,j}\, R^{0,i}_{i,0}$$

$$= \sum_{i,j\geq 0} g_{i,j}(x,q)\, x^{i-2j}, \text{ where } g_{i,j}(x,q) \in \mathbb{Z}[q,q^{-1},x].$$

Figure eight knot (after the extension)



$$\xrightarrow{F_K} \operatorname{tr}^q_{V\otimes V}(\beta) = \sum_{i\geq 0, j<0} (R^{-1})^{i,j}_{i,j} R^{i,0}_{0,i} (R^{-1})^{0,j}_{0,j} R^{0,i}_{i,0}$$

$$= 1 + 2\,x + (q^{-1} + 3 + q)\,x^2 + (2q^{-2} + 2q^{-1} + 5 + 2q + 2q^2)\,x^3 + \dots$$

The class of nice knots

Definition: A knot K is *nice* if there exists a braid representative of K admitting a $\{\pm\}$-coloring of its segments such that $\mathrm{tr}^q(\beta) \in \mathbb{Z}[q, q^{-1}][[x]]$.

In that case, $F_K(x, q) := \mathrm{tr}^q(\beta)$.

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Motivation: $\Delta_K(x)$ is **monic** whenever K is a fibered knot. Therefore,

$$\frac{1}{\Delta_K(x)} \in \mathbb{Z}[[x]] \subsetneq \mathbb{Q}[[x]].$$

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Theorem (Orland–S.–Saunder-A'Court–Svoboda)

All fibered knots up to 12 crossings are nice.

The F_K series and fiberedness

What do these series have in common?

(Besides coming from fibered knots)

$$\begin{aligned} F_{4_1}(x,q) &= x + 2\,x^2 + (q^{-1} + 3 + q)\,x^3 \\ &+ (2q^{-2} + 2q^{-1} + 5 + 2q + 2q^2)\,x^4 \\ &+ (q^{-4} + 3q^{-3} + 4q^{-2} + 5q^{-1} + 8 + 5q + 4q^2 + 3q^3 + q^4)\,x^5 + \dots \end{aligned}$$

$$\begin{aligned} F_{8_{20}}(x,q) &= x^2 + x^3 + (1 - q - q^3)\,x^4 \\ &+ (-q - q^3 - q^4)\,x^5 + (-q^{-1} - q - q^3 - q^4 + q^6 + q^9)\,x^6 \\ &+ (-q^{-2} - q^{-1} - q^3 - q^4 + q^5 + q^6 + q^7 + q^9 + q^{10} + q^{11})\,x^7 + \dots \end{aligned}$$

$$\begin{aligned} F_{10_{153}}(x,q) &= q^{-1}x^3 + (-q^{-3} + q^{-1})\,x^4 + (q^{-6} - q^{-3} + q^{-1})\,x^5 \\ &+ (-q^{-10} + q^{-6} - 2q^{-3})\,x^6 \\ &+ (q^{-15} - q^{-10} + q^{-7} + q^{-6} + q^{-5} - 2q^{-3} - 1)\,x^7 + \dots \end{aligned}$$

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$$F_{4_1}(x,q) = q^{g(4_1)-\lambda(4_1)}x^{g(4_1)} + 2\,x^2 + (q^{-1}+3+q)\,x^3 + (2q^{-2}+2q^{-1}+5+2q+2q^2)\,x^4 + (q^{-4}+3q^{-3}+4q^{-2}+5q^{-1}+8+5q+4q^2+3q^3+q^4)\,x^5 + \dots$$

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Fiberedness and Quantum Knot Invariants

Let K be a fibered knot. Denote by $g(K)$ its genus and by $\lambda(K)$ the Hopf invariant associated to the fiber surface of K.

Theorem (Orland–S.–Saunder-A'Court–Svoboda)

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1. The leading term in x is of the form $q^{\ell(K)}$ for some $\ell(K)$.
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Conjecture (Orland–S.–Saunder-A'Court–Svoboda)

$$\{\text{fibered knots}\} = \{\text{nice knots}\}.$$

From sign diagrams to the Hopf invariant

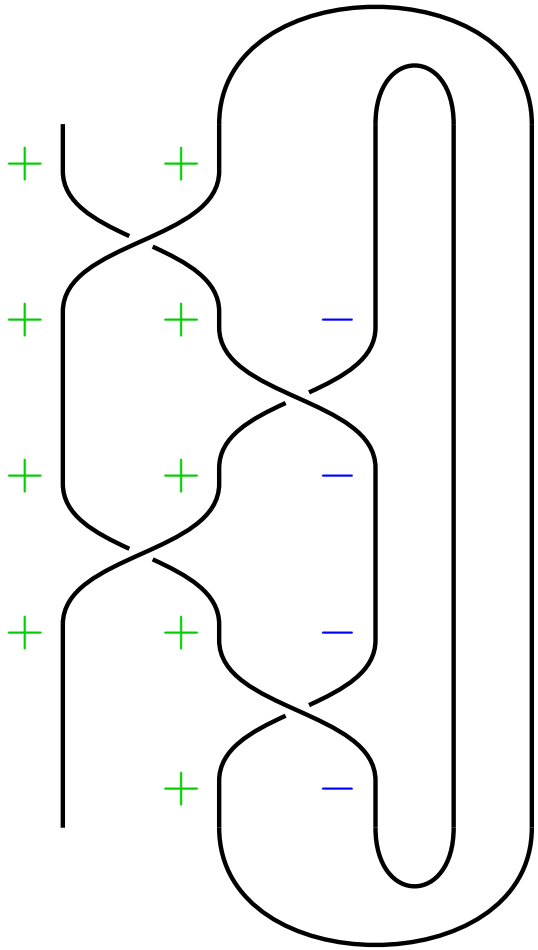
Sign diagrams contain information about the fibered surface of the knot!

Given a braid and a sign diagram, assign to each of its crossings:

	sign patterns	q-contr.	x-contr.
Positive crossing	$\begin{matrix}+&+\\+&+\end{matrix}$, $\begin{matrix}-&+\\-&+\end{matrix}$	$+1/2$	$+1/2$
	$\begin{matrix}+&-\\-&+\end{matrix}$, $\begin{matrix}-&+\\+&-\end{matrix}$	0	0
	$\begin{matrix}-&-\\-&-\end{matrix}$	$+1/2$	$-1/2$
Negative crossing	$\begin{matrix}-&-\\-&-\end{matrix}$, $\begin{matrix}+&-\\+&-\end{matrix}$	$-1/2$	$+1/2$
	$\begin{matrix}+&-\\-&+\end{matrix}$, $\begin{matrix}-&+\\+&-\end{matrix}$	0	0
	$\begin{matrix}+&+\\+&+\end{matrix}$	$-1/2$	$-1/2$

The sum of the q-contributions and x-contributions (with some extra braid-dependent factors) give $g(K) - \lambda(K)$ and $g(K)$ respectively.

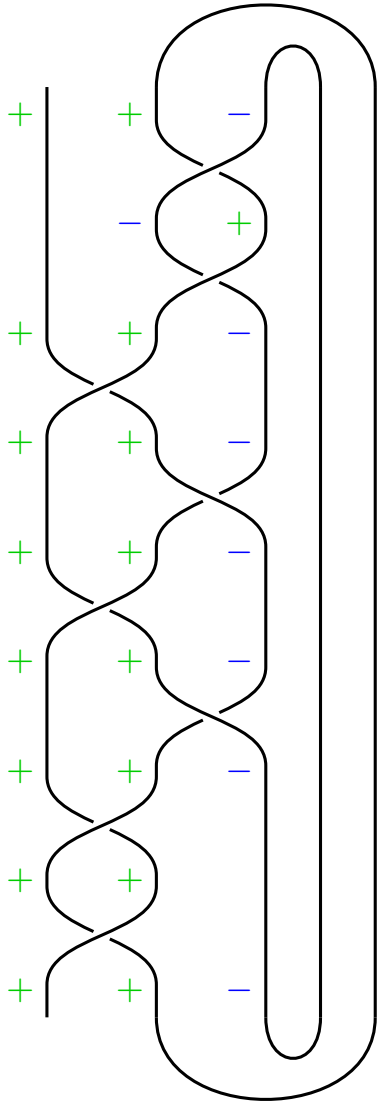
Example: figure-eight



	q-contr.	x-contr.
$\longmapsto$	$1/2$	$1/2$
$\longmapsto$	$-1/2$	$1/2$
$\longmapsto$	$1/2$	$1/2$
$\longmapsto$	$-1/2$	$1/2$

$$q\text{-deg} = \underbrace{0}_{\text{shift}} + 0 = 0 = g(4_1) - \lambda(4_1), \quad x\text{-deg} = \underbrace{-1}_{\text{shift}} + 2 = 1 = g(4_1)$$

Example: 8_{21}



	q-contr.	x-contr.
$\longmapsto$	0	0
$\longmapsto$	0	0
$\longmapsto$	$1/2$	$1/2$
$\longmapsto$	$-1/2$	$1/2$
$\longmapsto$	$1/2$	$1/2$
$\longmapsto$	$-1/2$	$1/2$
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$$q\text{-deg} = \underbrace{0}_{\text{shift}} + 1 = 1 = g(8_{21}) - \lambda(8_{21}), \quad x\text{-deg} = \underbrace{-1}_{\text{shift}} + 3 = 2 = g(8_{21})$$

Beyond fibered knots

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- For fibered knots:
  - $F_K(x,1) \overset{x\sim 0}{=\!=} \frac{1}{\Delta_K(x)} \in \mathbb{Z}[[x]]$
  - F_K has finite q-polynomials as x-coefficients.
- Now, for non-fibered knots:
  - $F_K(x,1) \overset{x\sim 0}{=\!=} \frac{1}{\Delta_K(x)} \in \mathbb{Q}[[x]]$
  - In general, F_K will have infinite q-series as its x-coefficients!

There is a large class of knots for which we are able to make sense of the (now ill-defined) inverted state sum, including strongly quasipositive knots.

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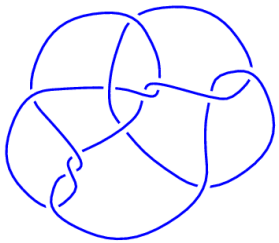
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Example:

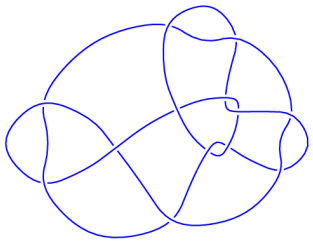
$$\begin{aligned} F_{9_{49}}(x,q) = {} & q^{-2}\left(-1+2q-q^2-3q^3+2q^4+2q^5+3q^6-3q^7-4q^8+\dots\right)x^2 \\ & + q^{-2}\left(-2+4q+q^2-8q^3-2q^4+4q^5+13q^6+4q^7-8q^8+\dots\right)x^3 \\ & + q^{-3}\left(1-5q+7q^2+7q^3-11q^4-14q^5-8q^6+25q^7+\dots\right)x^4 + O(x^5). \end{aligned}$$

What happens when a non-fibered knot has monic Alexander?

Two very different knots:



- Fibered, nice knot.
- $\widehat{HFK}(K,g) = x^3$
- $F_K = q^{-3}x^3 - 2q^{-3}x^4 + \left(q^{-5} + 4q^{-4} + 2q^{-2}\right)x^5 + O(x^6)$



- Non-fibered, not-nice knot.
- $\widehat{HFK}(K,g) = (2+m)x^3$
- $F_K = q^{-3}(1 - q^2 + 2q^4 - q^6 - q^7 + 2q^9 - q^{10} - q^{12} + \cdots)\,x^3 + q^{-2}(1 - 3q - 2q^2 + 4q^3 + 2q^4 - 2q^5 - 3q^6 + 5q^8 + \cdots)\,x^4 + q^{-4}(1 - 3q + 5q^2 - 3q^3 - 5q^4 + 7q^5 + 5q^6 - q^7 + \cdots)\,x^5 + O(x^6)$

Same Alexander polynomial:

$$\Delta_K(x) = \frac{1}{x^3} + \frac{1}{x^2} - \frac{5}{x} + 7 - 5x + x^2 + x^3$$

and same genus.

Figures: knotinfo.org.

$\ell(K)$ in knot Floer homology

Define $F_K(x,q) = q^{\ell(K)}x^{d(K)} +$ (higher order terms in q or x).

Conjecture: $d(K) = -\ell(K) = g(K)$.

Example:

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Conjecture: Let K be a fibered or strongly quasinegative. Define

$$\mu(K) := \min\{s \mid \widehat{HFK}_s(K, g(K)) \neq 0\}$$

Then, $d_K = g(K)$ and $\ell(K) = \mu(K)$.

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The previous conjecture is based on:

Observation: Let K be a strongly quasipositive knot of crossing number ≤ 16. Then, $\mu(K) = 0$.

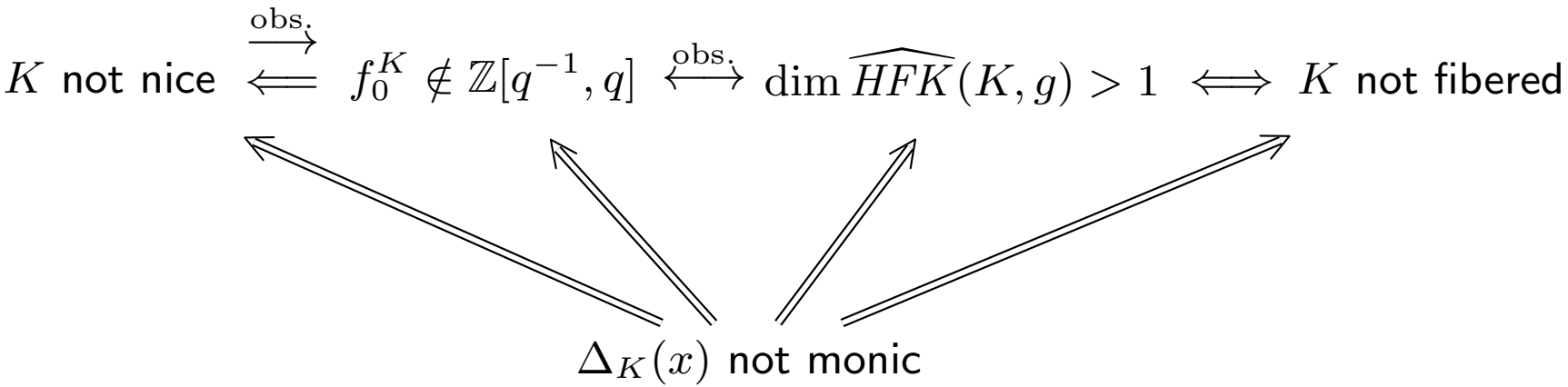
f_0^K seems to have greater distinguishing power than $\widehat{HFK}(K,g)$

$\pi\left(\widehat{HFK}(K,g)\right)$	$\pm q^{-\ell(K)} f_0^K(q)$	K
1	1	fibered knots
2	$1-q+q^3-q^6+q^{10}-q^{15}+q^{21}-\cdots$	$5_2, 7_3, 7_5, 9_{3,6,9}, 10_{128}$
3	$1-q+q^5-q^8+q^{16}-q^{21}+q^{33}+\ldots$	$7_2, 9_{47}, 12n_{581}$
3	$1-2q+q^2+3q^3-2q^4+\cdots$	$8_{15}, 9_{49}, 10_{49,66,80}$
4	$1-2q+q^2+2q^3-2q^4-q^6+\cdots$	$7_4, 9_{10,13,18,23}$
5	$1-3q+3q^2+4q^3-8q^4-2q^5+\cdots$	9_{38}
5	$1-2q+q^2+q^3-q^4+2q^5-2q^6+\cdots$	$10_{55,63}$
6	$1-2q+q^2+q^3+\cdots$	9_5
6	$1-3q+3q^2+3q^3-7q^4+\cdots$	10_{53}
7	$1-2q+q^2+3q^5-3q^6-q^7-q^8+\cdots$	9_{35}
7	$1-3q+3q^2+2q^3-6q^4+4q^5+\cdots$	10_{101}
8	$1-4q+6q^2+3q^3-17q^4+\cdots$	10_{120}
$1+2m$	$1+2q+q^2+q^3+2q^4+2q^5+2q^6+\cdots$	$12n_{801}$
$1+m$	$1+q+q^4+q^9+q^{16}+q^{25}+q^{36}+\cdots$	$12n_{293,321,750,830}$

where $F_K(x,q) = f_0^K(q)x^{d_K} + O(x^{d_K+1})$.

We observe that $\widehat{HFK}(K_1,g_1) \cong \widehat{HFK}(K_2,g_2) \Rightarrow f_0^{K_1}(q) = f_0^{K_2}(q)$.

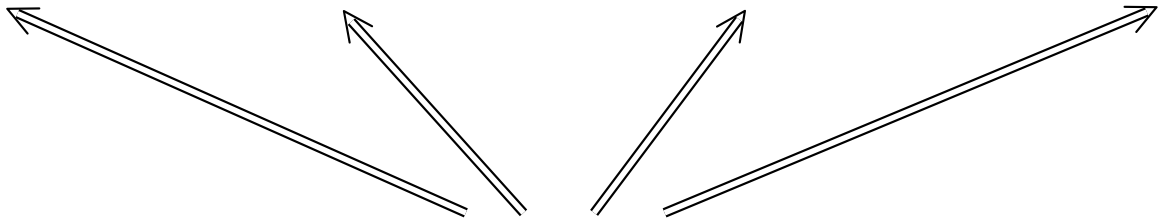
F_K and $\widehat{HFK}$



F_K and $\widehat{HFK}$

$$\min_q f_0^K \overset{\text{obs.}}{\longleftrightarrow} \min\{s \mid \dim \widehat{HFK}_s(K,g) \neq 0\}$$

$$K \text{ not nice} \overset{\text{obs.}}{\Longleftarrow} f_0^K \notin \mathbb{Z}[q^{-1},q] \overset{\text{obs.}}{\longleftrightarrow} \dim \widehat{HFK}(K,g) > 1 \iff K \text{ not fibered}$$



$$\Delta_K(x) \text{ not monic}$$

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Questions:

1. How does this extra structure read the fibered surface of a knot and how can we construct it?

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- Algebra: Use the rep. theory of $U_q(\mathfrak{sl}_2)$ to build knot invariants.
- The two seem to be related by adding some extra structure on a braid presentation.

Questions:

1. How does this extra structure read the fibered surface of a knot and how can we construct it?
2. What geometrical information is encoded in the rest of the F_K series? (infinite terms left to study).

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Thank you!

Appendix slides

Hopf invariant or enhanced Milnor number

Alt. formal definition: Let ξ_1, ξ_2 be two no-where zero vector fields in general position in S^3. Let

$$\Delta(\xi_1, \xi_2) := \{p \in S^3 \mid \xi_1(p) = t\,\xi_2(p) \text{ for some } t > 0\}.$$

Define the enhanced Milnor invariant or *Hopf invariant* of ξ_K as

$$\lambda(\xi_K) \equiv \lambda(K) := lk\left(\Delta(\xi_K, \xi_{std}), \Delta(\xi_K, -\xi_{std})\right)$$

where ξ_{std} is the canonical contact structure of S^3 and lk denotes the linking number.

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Representation theory at generic q

$$[H,E]=2E,\quad [H,F]=2F,\quad [E,F]=\frac{K-K^{-1}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}},\quad K^{\pm\frac{1}{2}}=q^{\pm\frac{H}{4}}$$

$$R=q^{\frac{H\otimes H}{4}}\sum_{k\geq 0}\frac{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^k}{[k]_q}q^{k(k-1)/4}E^k\otimes F^k\in U_q(\mathfrak{sl}_2)\hat{\otimes}U_q(\mathfrak{sl}_2)$$

Semisimple invariants

For generic $q\in\mathbb{C}$: for every $n\in\mathbb{N}$ there is a simple $U_q(\mathfrak{sl}_2)$ representation V_n of dimension $n+1$.

$$E\cdot v_j=q^{\frac{n-2j}{4}}[j]_q\,v_{j-1},\quad F\cdot v_j=q^{-\frac{n-2j-2}{4}}[n-j]_q\,v_{j+1},\quad K\cdot v_j=q^{\frac{n-2j}{2}}v_j$$

$j=0,\dots,n$.

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New R-matrix formulas

extended R-matrix	condition
$R^{i',j'}_{i,j} = \delta^{i'+j'}_{i+j} q^{jj'} (qx)^{\frac{j+j'+1}{2}} \begin{bmatrix} i \\ i-j' \end{bmatrix}_q (q^{j+1}x;q)_{i-j'}$	$i \geq j' \geq 0$ $0 > i \geq j'$
$R^{i',j'}_{i,j} = \delta^{i'+j'}_{i+j} q^{jj'} (qx)^{\frac{j+j'+1}{2}} \begin{bmatrix} i \\ j' \end{bmatrix}_q \frac{1}{(q^j x;q^{-1})_{j'-i}}$	$j' \geq 0 > i$
$(R^{-1})^{i',j'}_{i,j} = \delta^{i'+j'}_{i+j} q^{-ii'} (qx)^{-\frac{i+i'+1}{2}} \begin{bmatrix} j \\ j-i' \end{bmatrix}_{q^{-1}} (q^{-i-1}x^{-1};q^{-1})_{j-i'}$	$j \geq i' \geq 0$ $0 > j \geq i'$
$(R^{-1})^{i',j'}_{i,j} = \delta^{i'+j'}_{i+j} q^{-ii'} (qx)^{-\frac{i+i'+1}{2}} \begin{bmatrix} j \\ i' \end{bmatrix}_{q^{-1}} \frac{1}{(q^{-i}x^{-1};q)_{i'-j}}$	$i' \geq 0 > j$

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Habiro series

Habiro coefficients $a_n(q) \in \mathbb{Z}[q^{\pm 1}]$, $n \geq 0$. Habiro series:

$$H_K(y,q) = \sum_{n=0}^{\infty} a_n(q)\sigma_n(y,q) \in \Lambda = \varprojlim_n \mathbb{Z}[q^{\pm 1}][y]/(\sigma_n) \tag{5.1}$$

where

$$H_K(q^n + q^{-n} - 2, q) = J_n(q) \qquad \forall n \geq 0. \tag{5.2}$$

Remark: $\{J_n\} \leftrightarrow \{a_n\}$.

Theorem (Garoufalidis–Bar-Natan, Rozansky, Habiro): There exist polynomials $\{P_K^{(j)}(y)\}_{j\geq 0} \subseteq \mathbb{Z}[y]$ such that

$$H_K(y, 1+h) = \frac{1}{\Delta_K(x)} + \sum_{i=1}^{\infty} \frac{P_K^{(j)}(x)}{\Delta_K(x)^{2j+1}} h^j \tag{5.3}$$

where the equality holds when expanding at $h = 0$ and $y = x + x^{-1} - 2 = 0$.

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