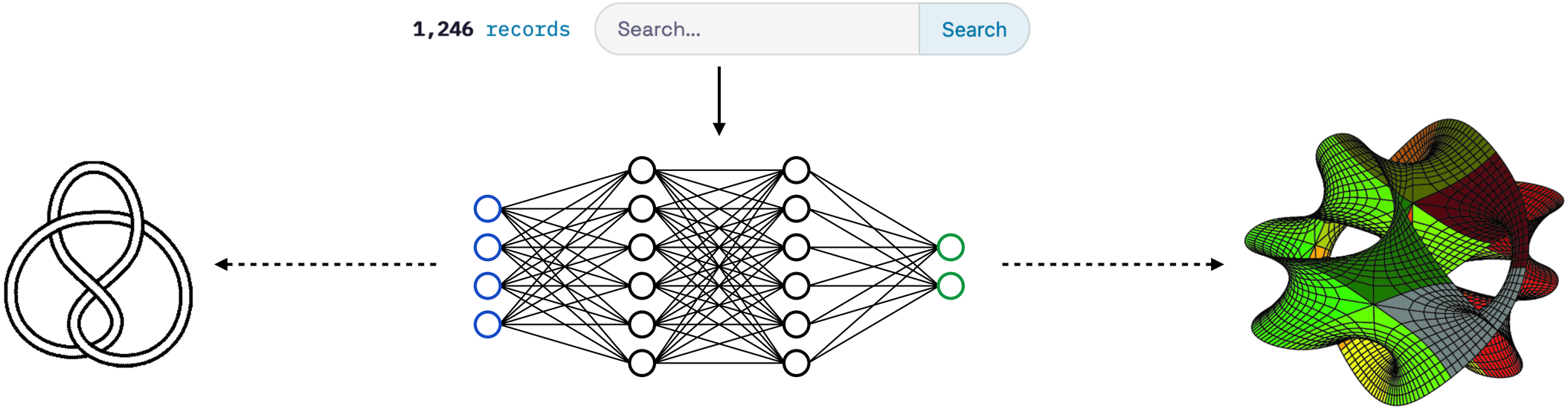
The first large-scale computation of the Gukov—Manolescu series

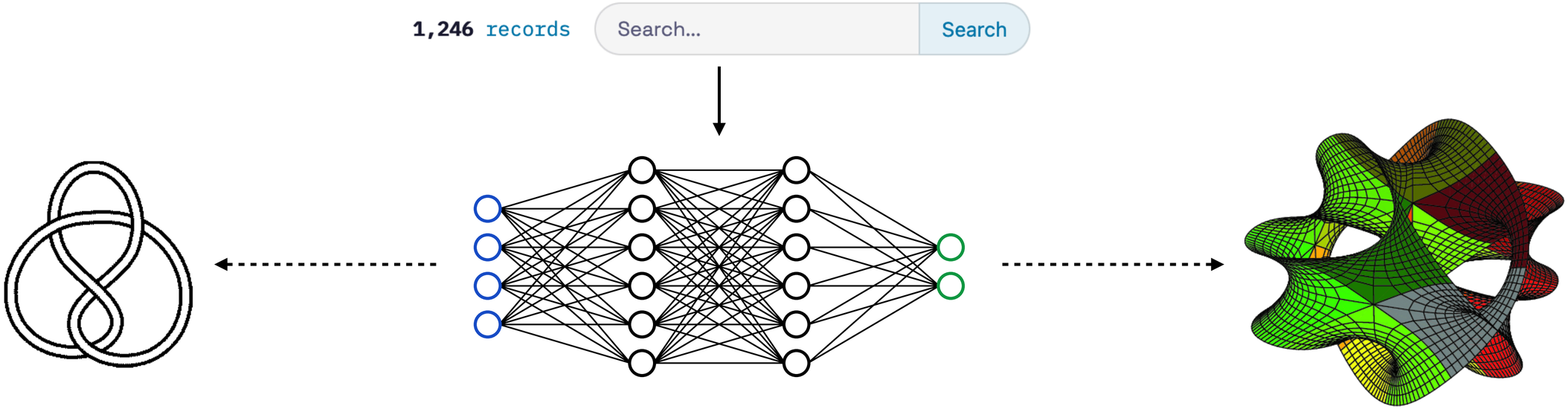


String Data 2025

Lara San Martín Suárez (Caltech)

Images: victorzhou.com, katlas.org, Hanson—Sha.

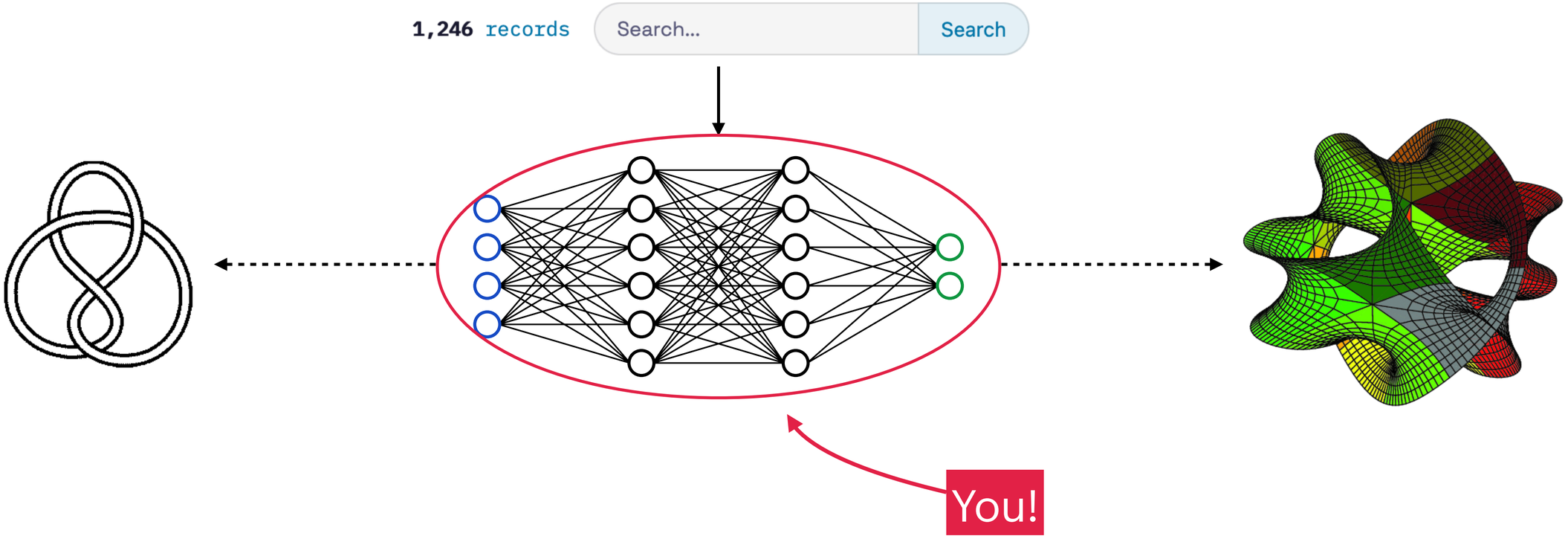
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Advisor

Collaborators



S. Gukov



P. Orland



D. Passaro



T. Saunders-A'Court



J. Svoboda

Today I'll be explaining

- Review of quantum invariants and construction of the GM series.
- Algorithm implemented to construct the database.
- How you can access and re-compute our data.
- First results obtained through human learning.

1990's

Polynomial invariants

Knots 

Quantum knot invariants

3-dim. information

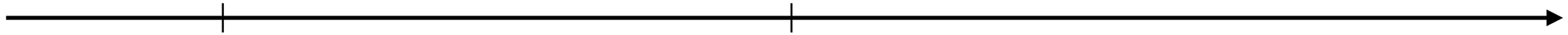
1990's

Polynomial invariants

2000's

Homological invariants

Knots



Quantum knot invariants

3-dim. information

Khovanov, knot Floer

4-dim. information

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Quantum knot invariants

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Khovanov, knot Floer

4-dim. information

3-manifolds



WRT invariants

SU(2) Chern-Simons theory

	1990's	2000's
	Polynomial invariants	Homological invariants
Knots	Quantum knot invariants	Khovanov, knot Floer
	3-dim. information	4-dim. information
3-manifolds	WRT invariants	Is there a homology theory for WRT?
	SU(2) Chern-Simons theory	

Image: vecteezy.com

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2015 – 2020's

Power series invariants

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2016

3-manifolds

WRT invariants

SU(2) Chern-Simons theory

Is there a homology theory for WRT?

$\hat{Z}$



Counts of BPS states

Gukov-Pei-Putrov-Vafa

Image: vecteezy.com

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GM series (F_K)

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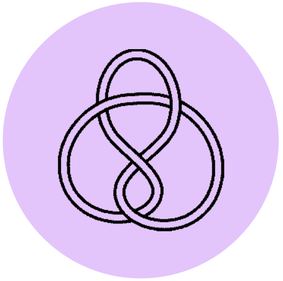


Counts of BPS states

Gukov-Pei-Putrov-Vafa

Image: katlas.org

Knot complement



$S^3 \backslash K$

Dehn surgery

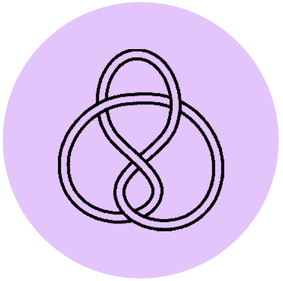
3-manifold M_3

Theorem (Lickorish, Wallace):

Every connected, oriented, closed 3-manifold arises by performing Dehn surgery along a link $K \subset S^3$.

Image: katlas.org, vecteezy.com

Knot complement



$S^3 \backslash K$

$$F_K = x^{1/2} \sum_{n=0}^{\infty} f_n(q)\, x^n$$

Dehn surgery

surgery formula

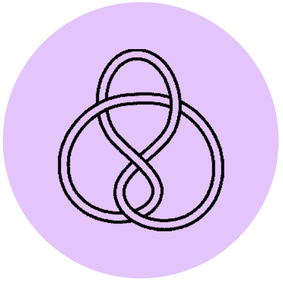
$$\oint_{|x|=1} \frac{dx}{2\pi i x} \cdots$$

3-manifold M_3

$$\hat{Z}(M_3) = q^{\Delta_b} \left(\sum_{n=0}^{\infty} a_n\, q^n \right)$$

Image: katlas.org, vecteezy.com

Knot complement



$S^3 \backslash K$

$$F_K = x^{1/2} \sum_{n=0}^{\infty} f_n(q)\, x^n$$

surgery formula

$$\oint_{|x|=1} \frac{dx}{2\pi i x} \ldots$$



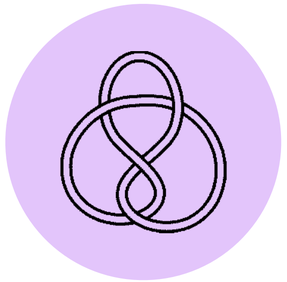
Dehn surgery

3-manifold M_3

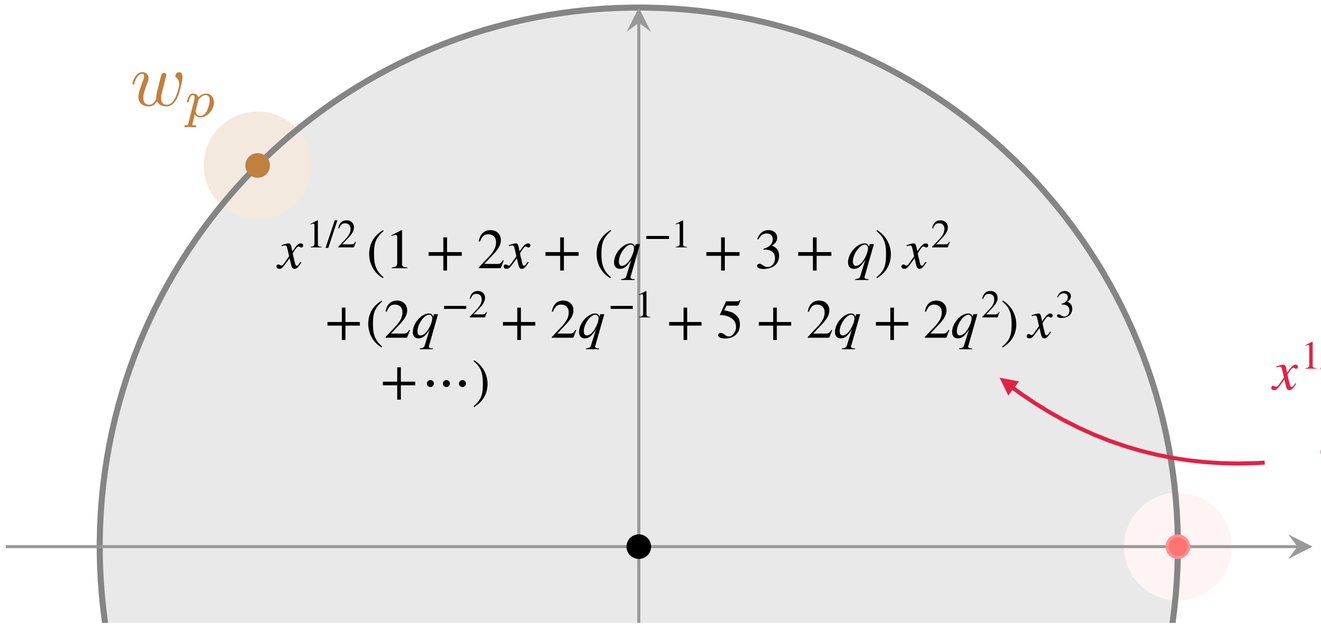
$$\hat{Z}(M_3) = q^{\Delta_b} \left(\sum_{n=0}^{\infty} a_n\, q^n \right)$$



Image: katlas.org



Gukov-Manolescu, 2018: resurgent analysis at $q = 1$

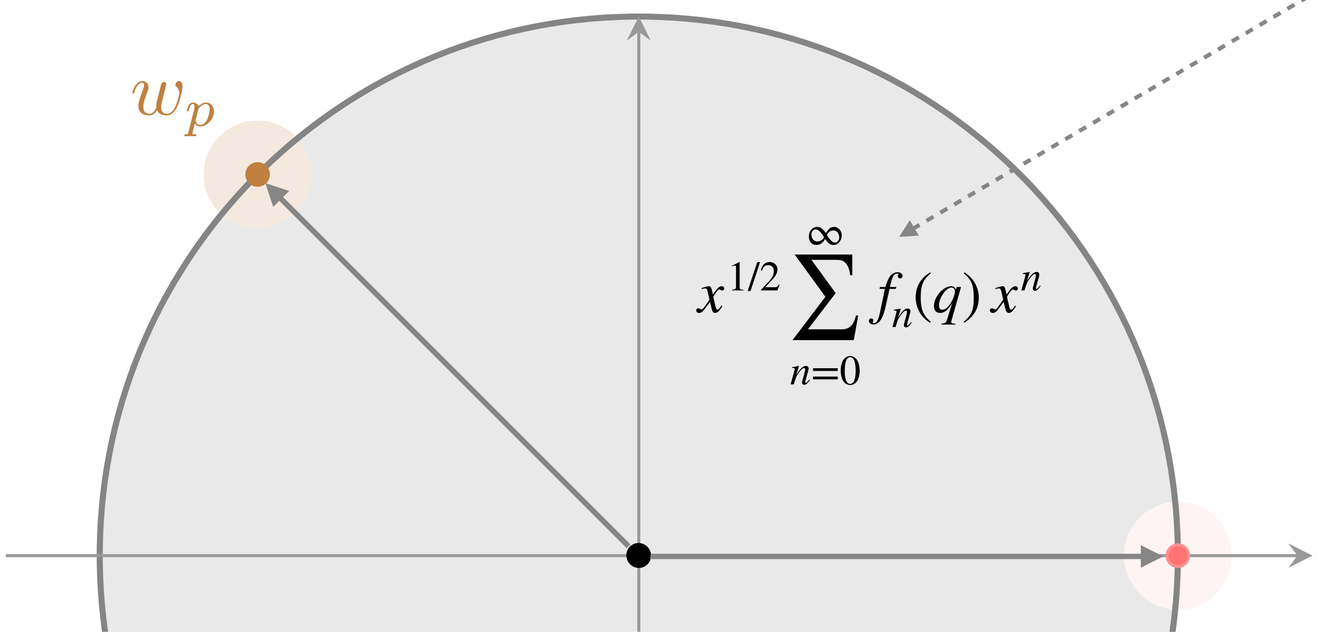


$$x^{1/2}(1 + 2x + (5 + \hbar^2 - \hbar^3 + \hbar^4 + \cdots)x^2 + (13 + 10\hbar^2 - 10\hbar^3 + 12\hbar^4 + \cdots)x^3 + \cdots)$$

$$q = 1 + \hbar$$

Gukov-Manolescu, 2018: resurgent analysis at $q = 1$

S. Park, 2020–21: quantum groups at generic q

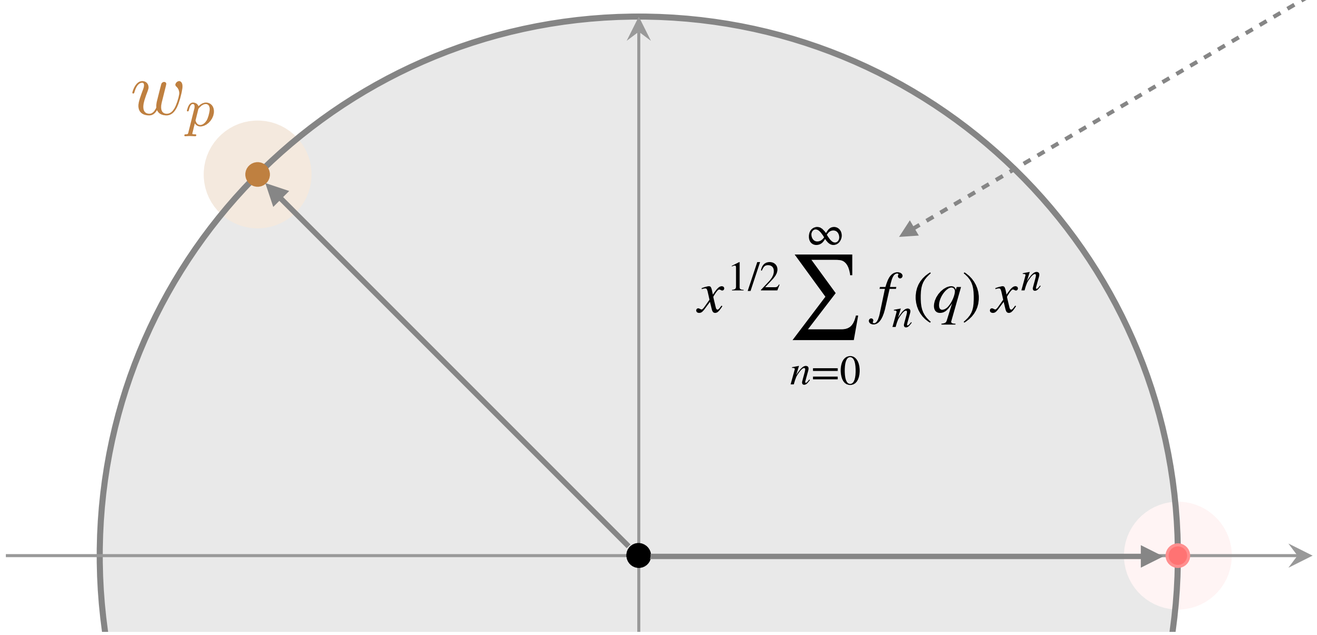


$$\frac{x^{1/2} - x^{-1/2}}{\Delta_K(x)} + \sum_{j=1}^{\infty} \hbar^j \frac{P_K^{(j)}(x)}{\Delta_K^{2j+1}(x)} \qquad q = 1 + \hbar$$

Alexander polynomial: $\Delta\left(\times\right) - \Delta\left(\times\right) = \left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)\Delta\left(\times\right)$ $\qquad \Delta\left(\bigcirc\right) = 1$

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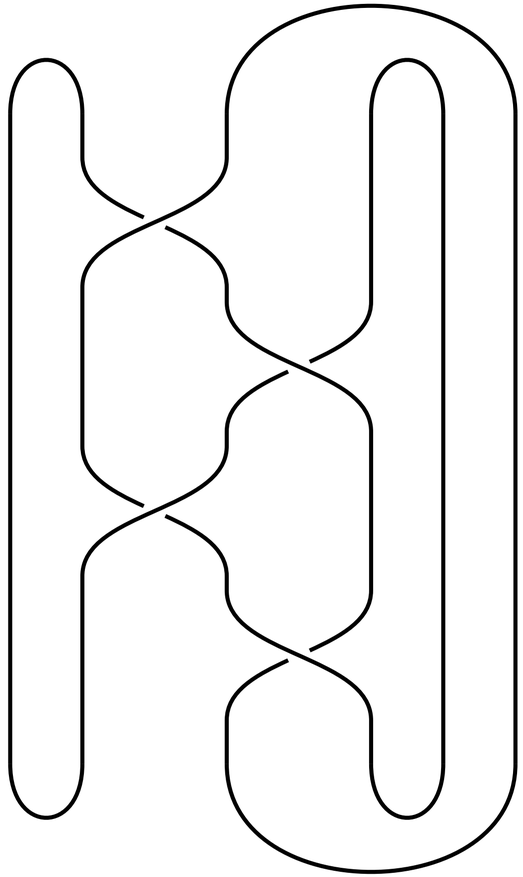


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Reshetikhin—Turaev: state sum over finite-dimensional modules of $U_q(\mathfrak{sl}_2)$ at generic q

1990's:

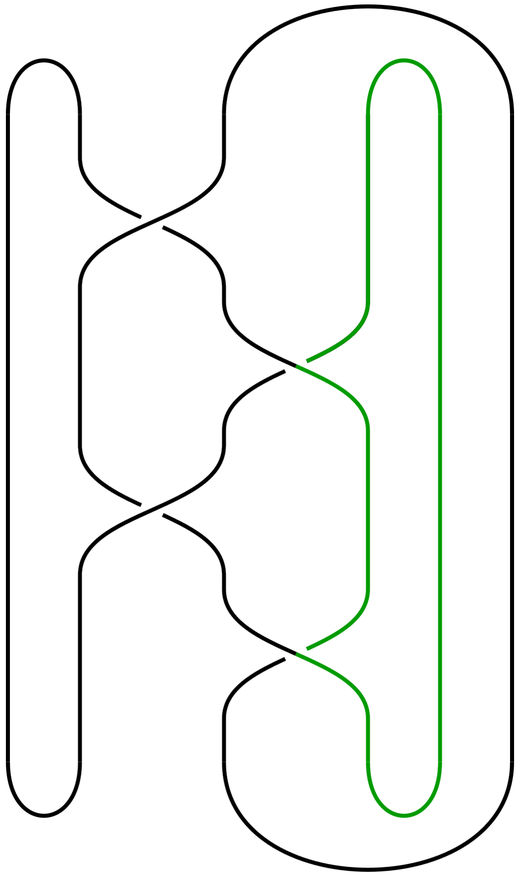


Quantum trace over a finite-dimensional vector space

$$V_n(-n) \underset{f}{\overset{e}{\rightleftharpoons}} V_n(-n+2) \underset{f}{\overset{e}{\rightleftharpoons}} \cdots \underset{f}{\overset{e}{\rightleftharpoons}} V_n(n-2) \underset{f}{\overset{e}{\rightleftharpoons}} V_n(n)$$

Park: state sum over (infinite-dimensional) Verma modules of $U_q(\mathfrak{sl}_2)$ at generic q

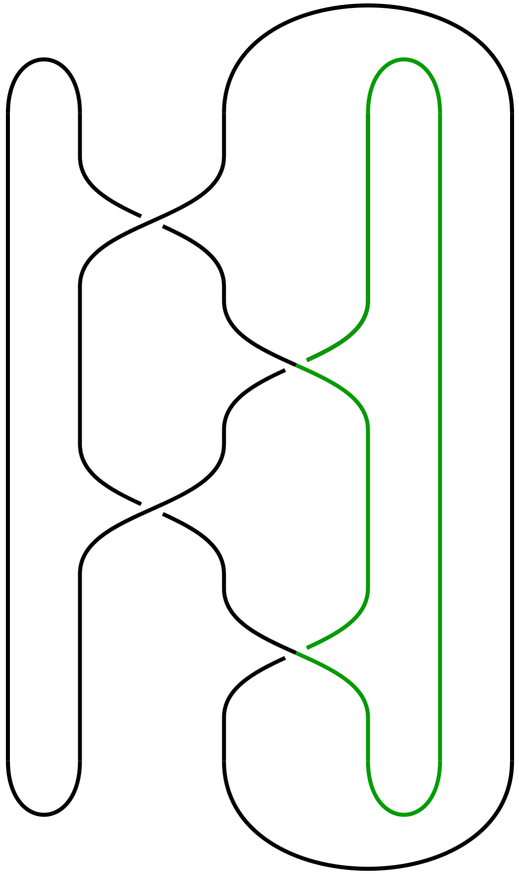
2020's:



Modified quantum trace over an infinite-dimensional vector space dependent on the choice of a (simple) multicycle on the braid

Park: state sum over (infinite-dimensional) Verma modules of $U_q(\mathfrak{sl}_2)$ at generic q

2020's:

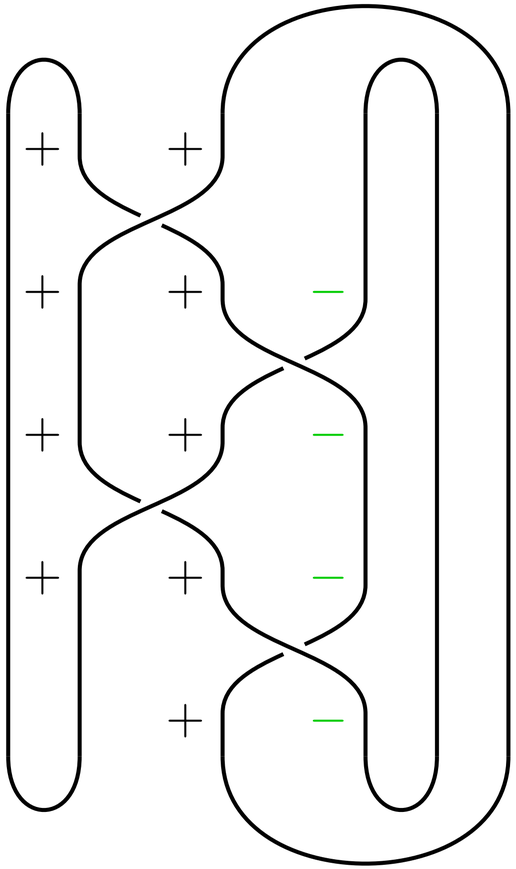


Modified quantum trace over an infinite-dimensional vector space dependent on the choice of a (simple) multicycle on the braid

$$\cdots \underset{f}{\overset{e}{\rightleftharpoons}} V_\infty(\lambda-2) \underset{f}{\overset{e}{\rightleftharpoons}} V_\infty(\lambda) \underset{f}{\overset{e}{\rightleftharpoons}} V_\infty(\lambda+2) \underset{f}{\overset{e}{\rightleftharpoons}} V_\infty(\lambda+4) \underset{f}{\overset{e}{\rightleftharpoons}} \cdots$$

Park: state sum over (infinite-dimensional) Verma modules of $U_q(\mathfrak{sl}_2)$ at generic q

2020's:



Modified quantum trace over an infinite-dimensional vector space dependent on the choice of a (simple) multicycle on the braid

Highest weight Verma $\cdots \underset{f}{\overset{e}{\rightleftharpoons}} \langle v_1 \rangle \underset{f}{\overset{e}{\rightleftharpoons}} \langle v_0 \rangle \underset{f}{\overset{e}{\rightleftharpoons}} \langle v_{-1} \rangle \underset{f}{\overset{e}{\rightleftharpoons}} \langle v_{-2} \rangle \underset{f}{\overset{e}{\rightleftharpoons}} \cdots$ Lowest weight Verma

1990's:

Braid ⇝ (many) one-variable polynomials.

No convergence issues.

2020's:

Braid & a choice of a multicycle ⇝ a two-variable infinite series.

Convergence depends on the choice of braid and multicycle.

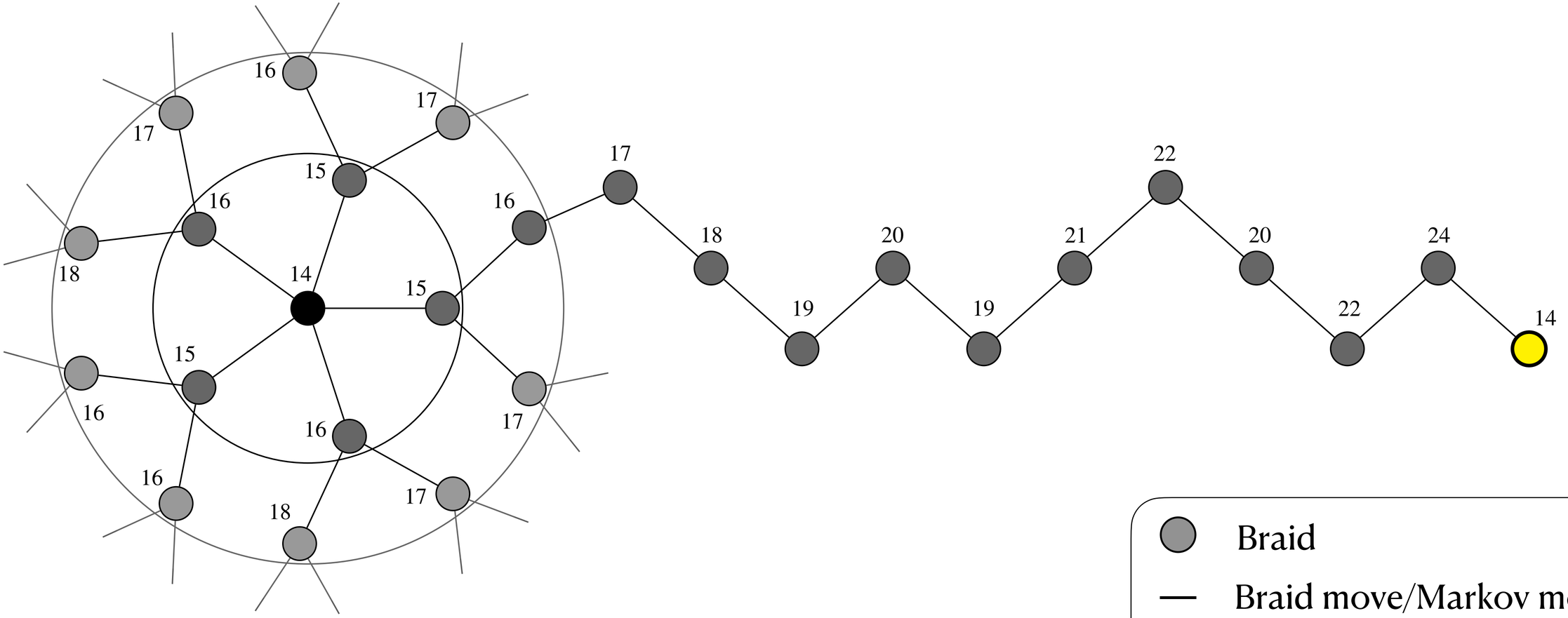
Why is this problem hard?

- We need a minimal length braid.

Why is this problem hard?

- We need a minimal length braid.
- Finding minimal length braids is hard.

Why is this problem hard?



Braid

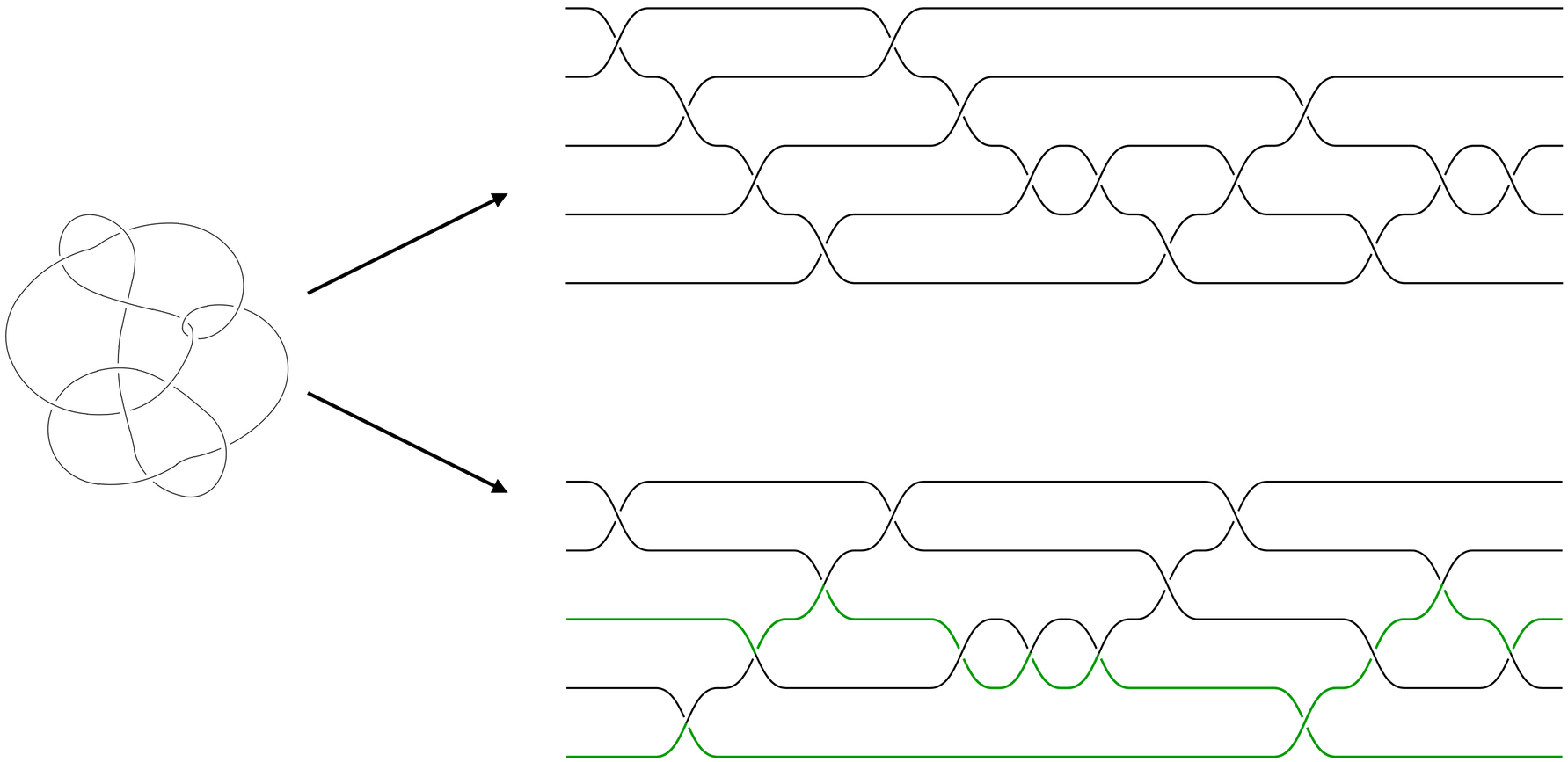
— Braid move/Markov move

14 Braid length

Why is this problem hard?

- We need a minimal length braid.
- Finding minimal length braids is hard.
- Finding a minimal length braid that admits a “convergent multicycle” is *very* hard.

Why is this problem hard?



$0/2^{14}$ multicycles is “convergent”.

$1/2^{14}$ multicycles is “convergent”.

Why is this problem hard?

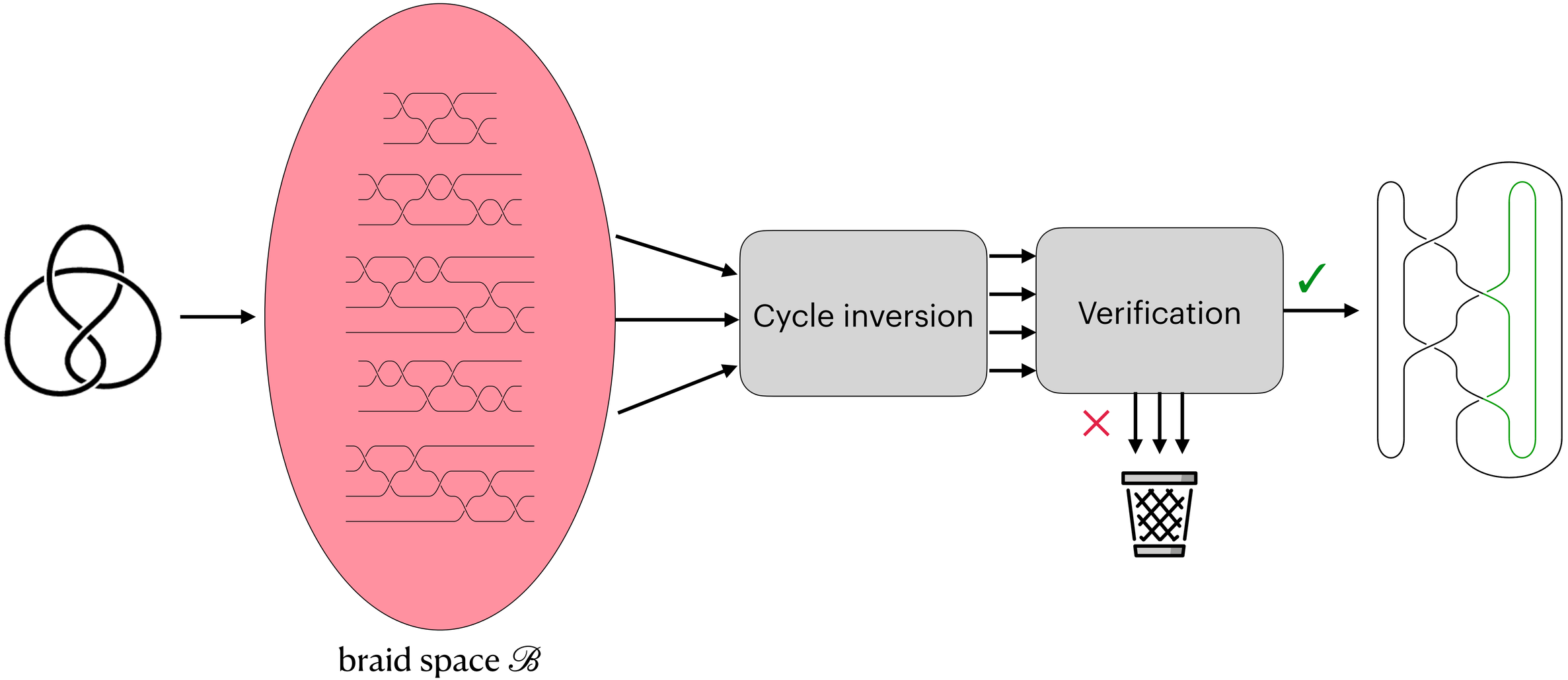
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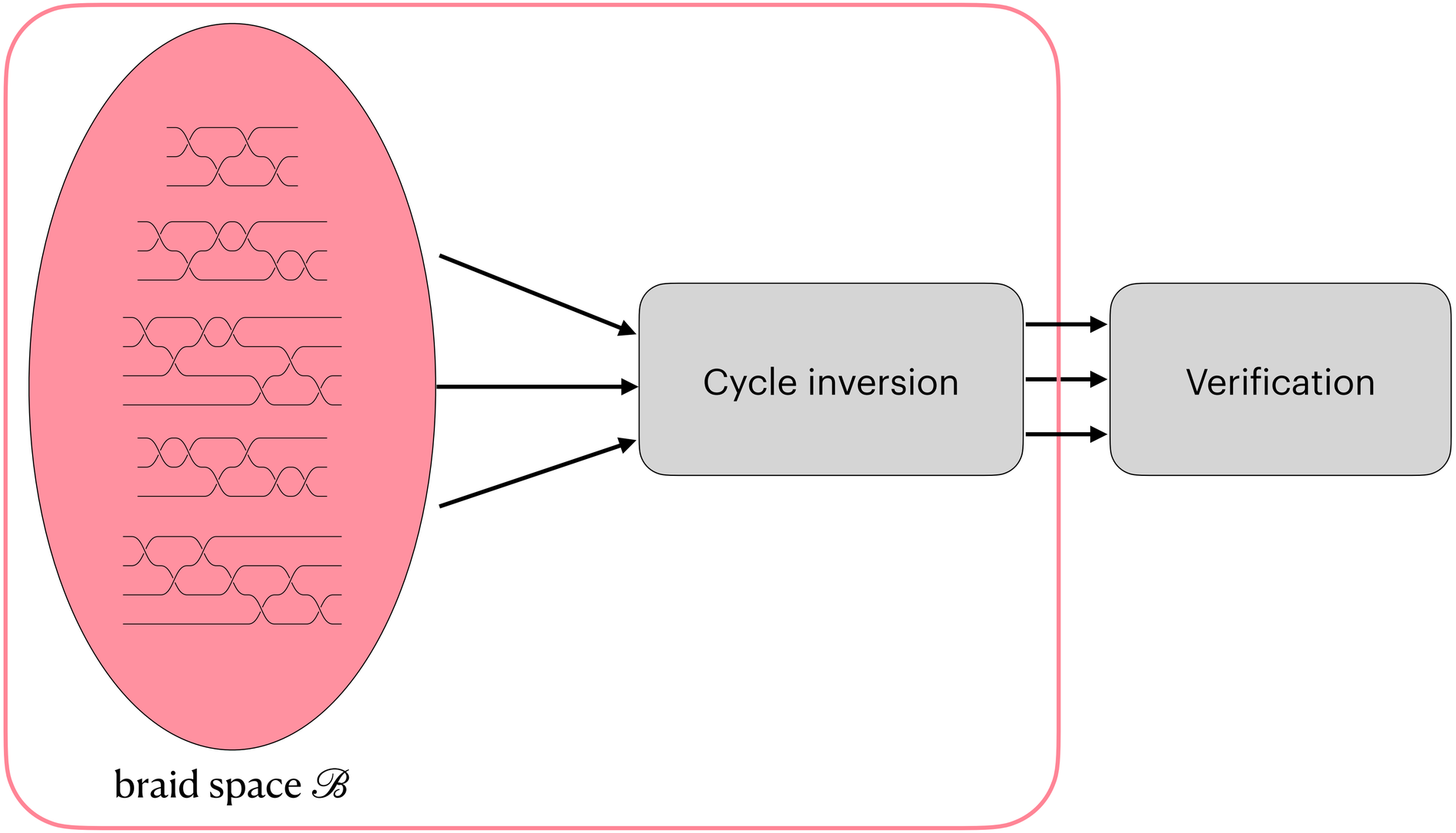


Side remark: we expect that this choice of multicycle has a geometrical meaning

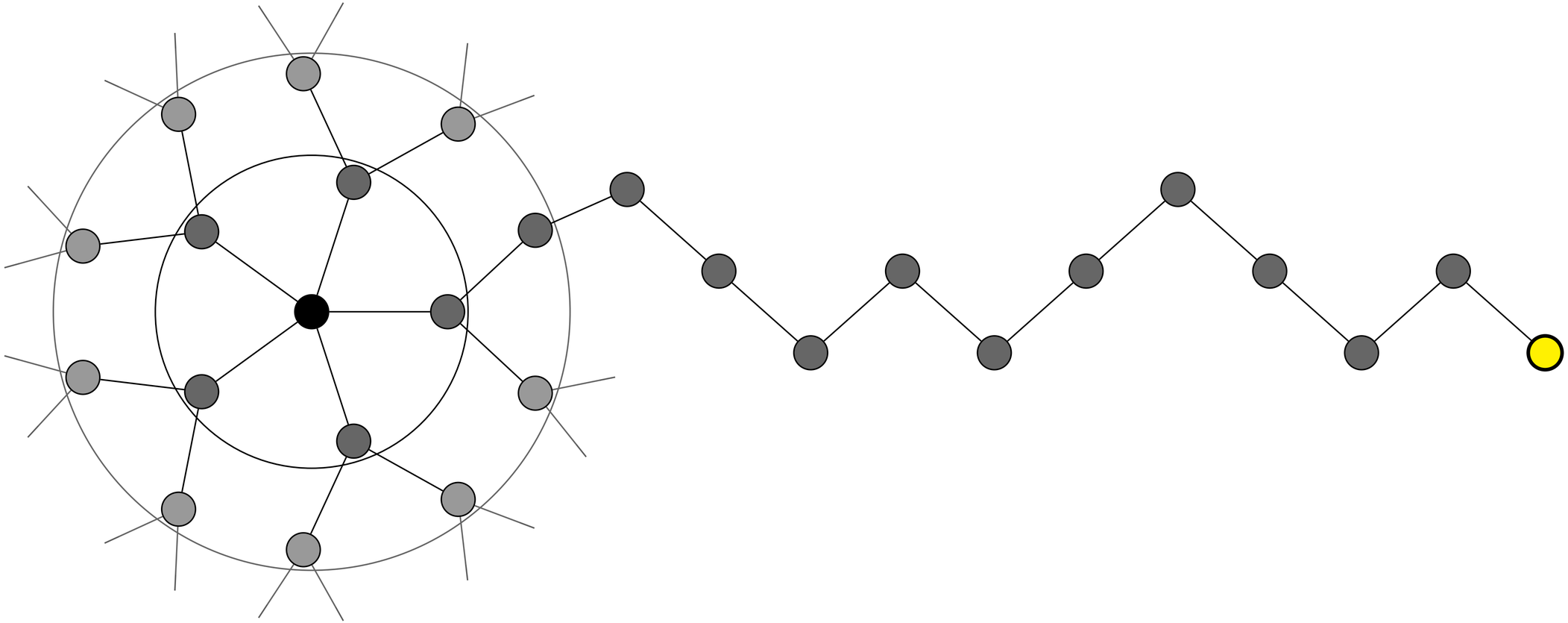
So... how do we do it?

Image: wikipedia.org.

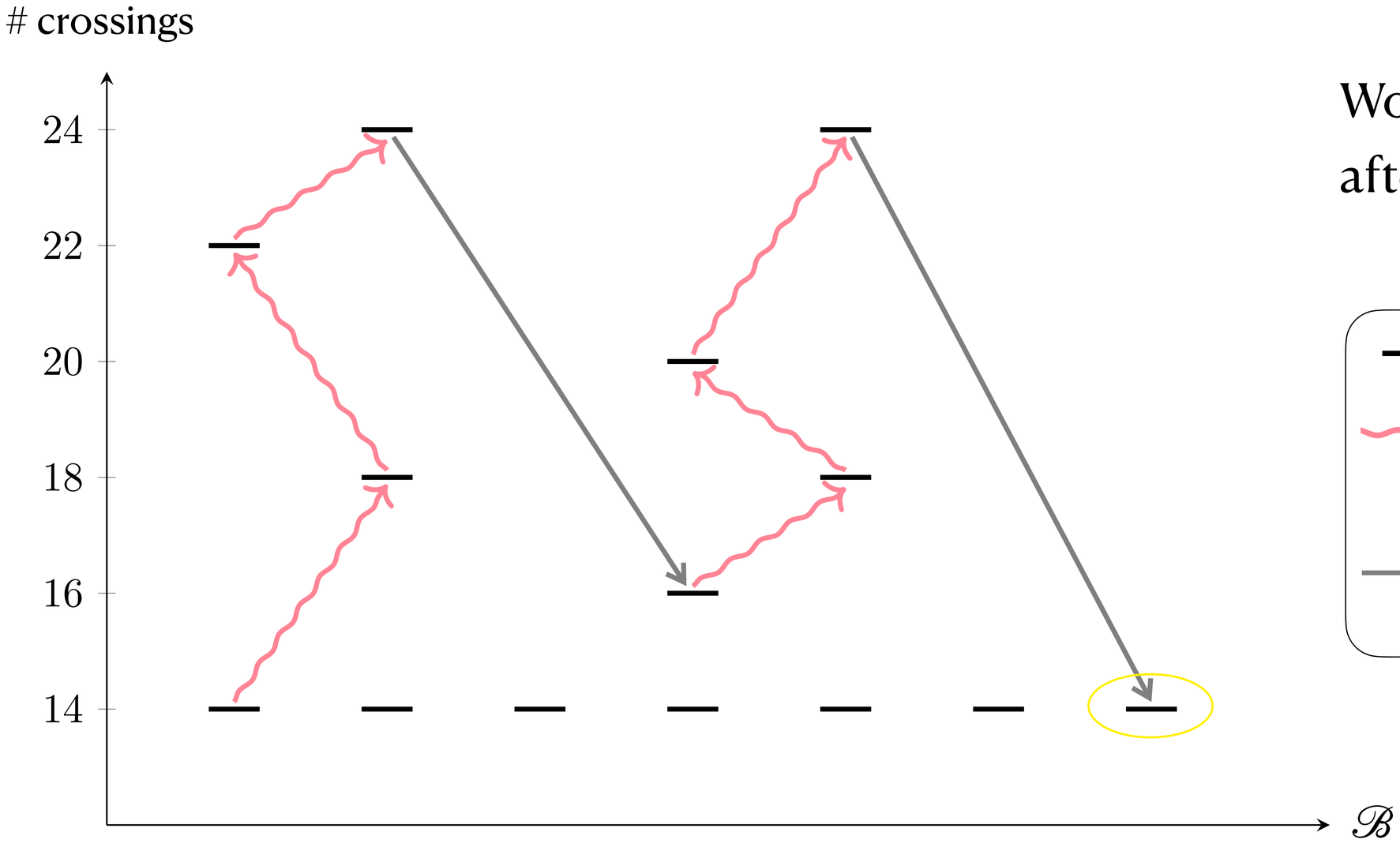



Cycle inversion
Verification
braid space $\mathscr{B}$

~~Search in~~ $\mathcal{B}$



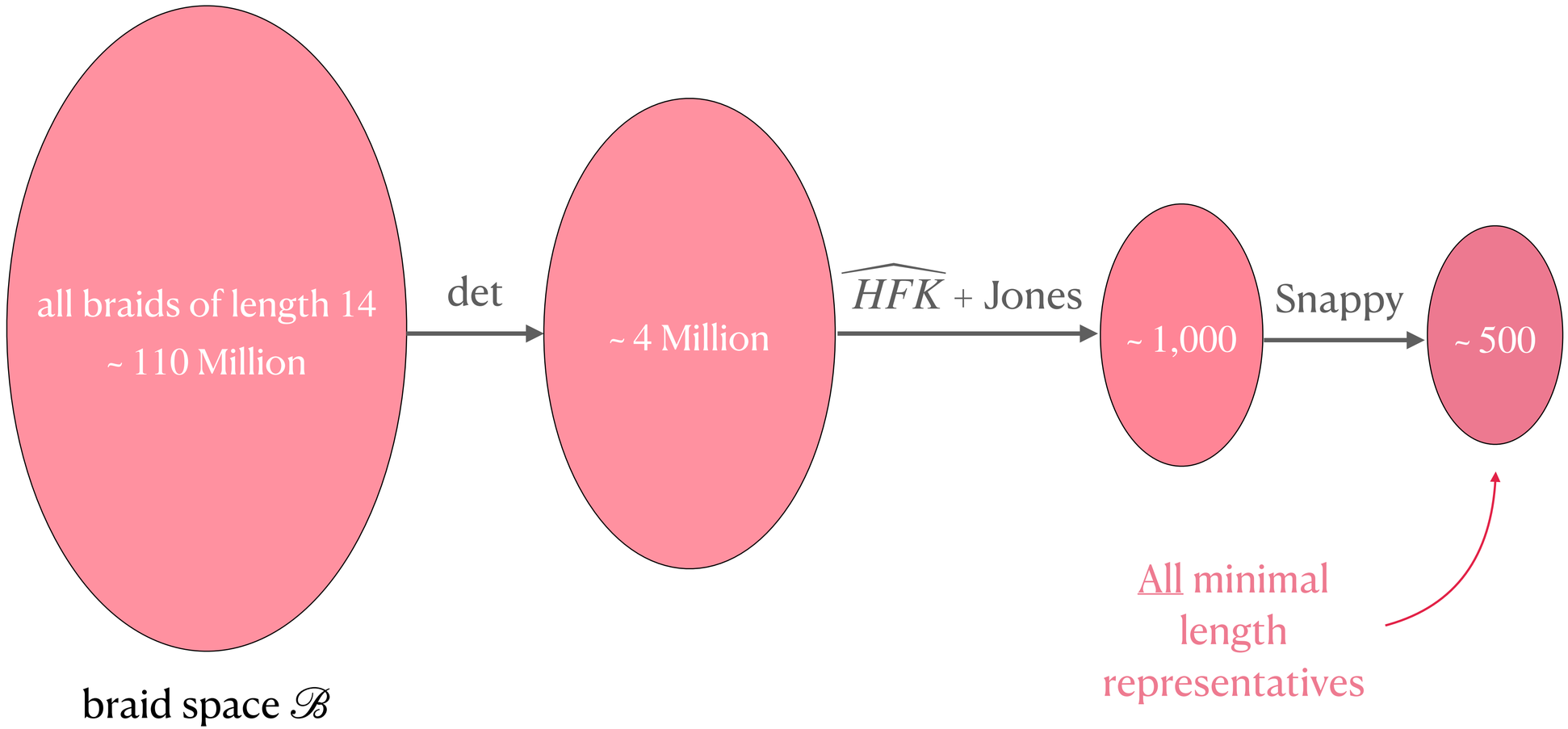
Random search in $\mathscr{B}$

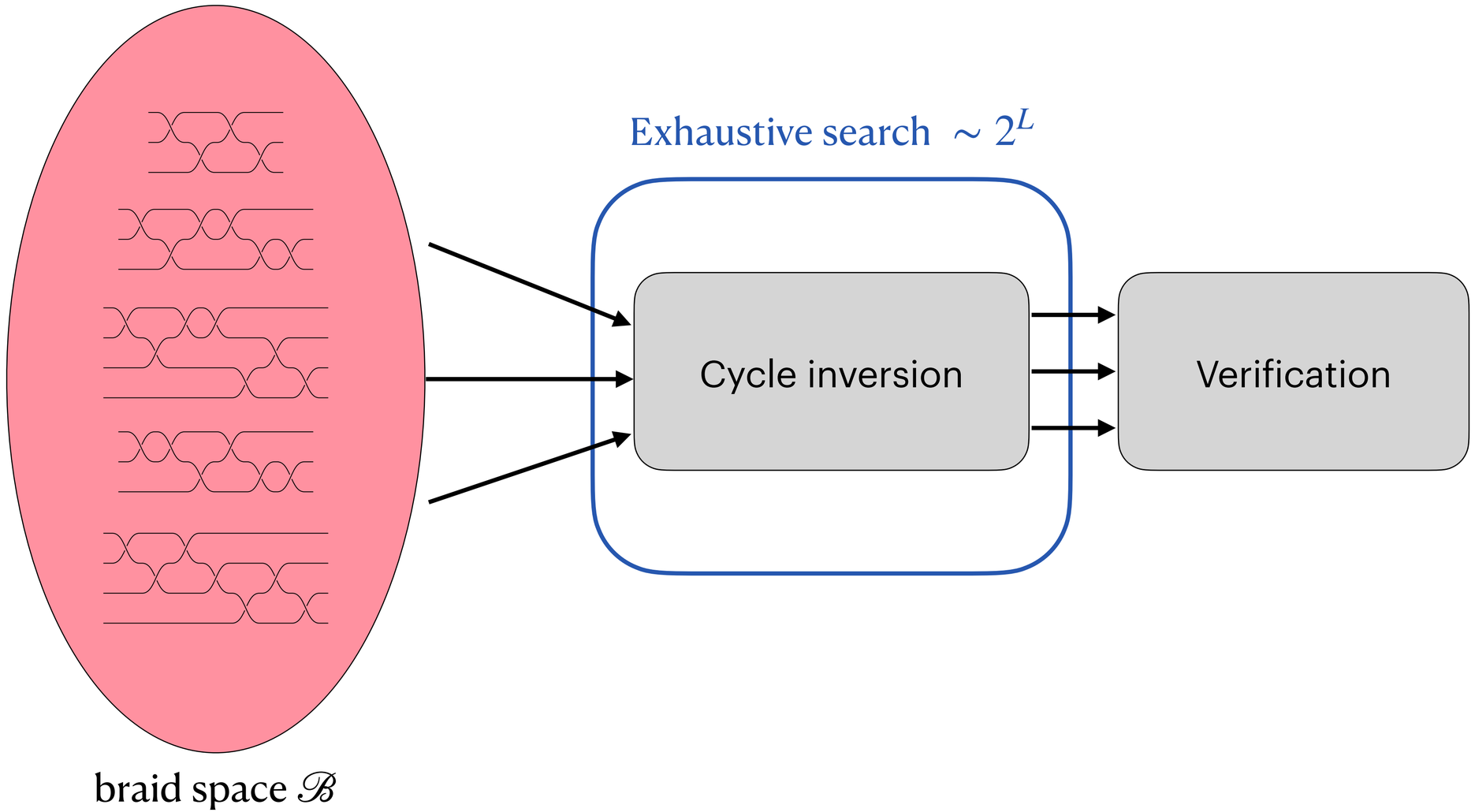


Works for 99.8% of knots after ~10^6 iterations.

- Braid
- Randomized sequence of ~10^3 braid/Markov moves
- Braid normalization procedure

For the remaining 0.2%:




Exhaustive search $\sim 2^L$
Cycle inversion
Verification
braid space $\mathscr{B}$

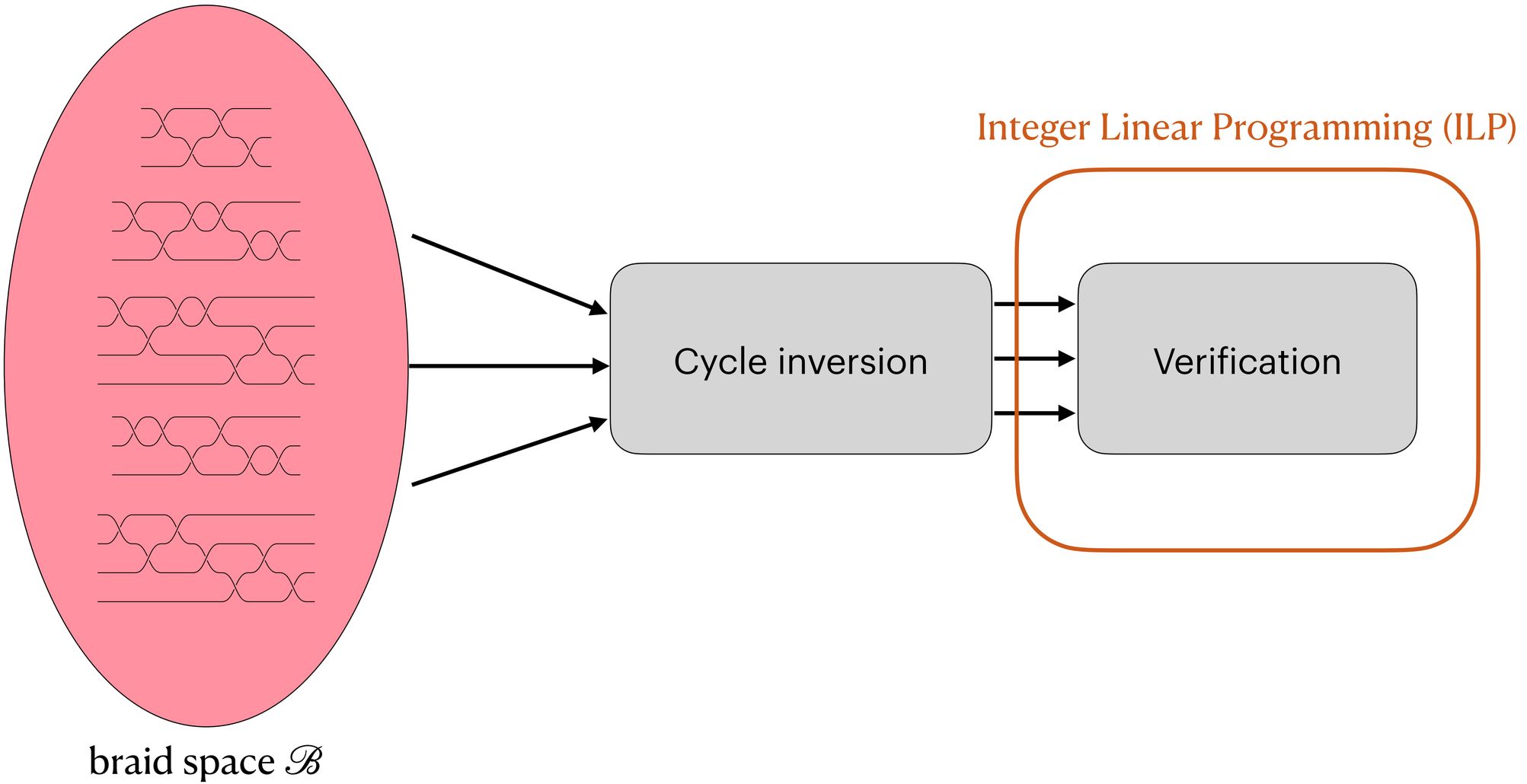
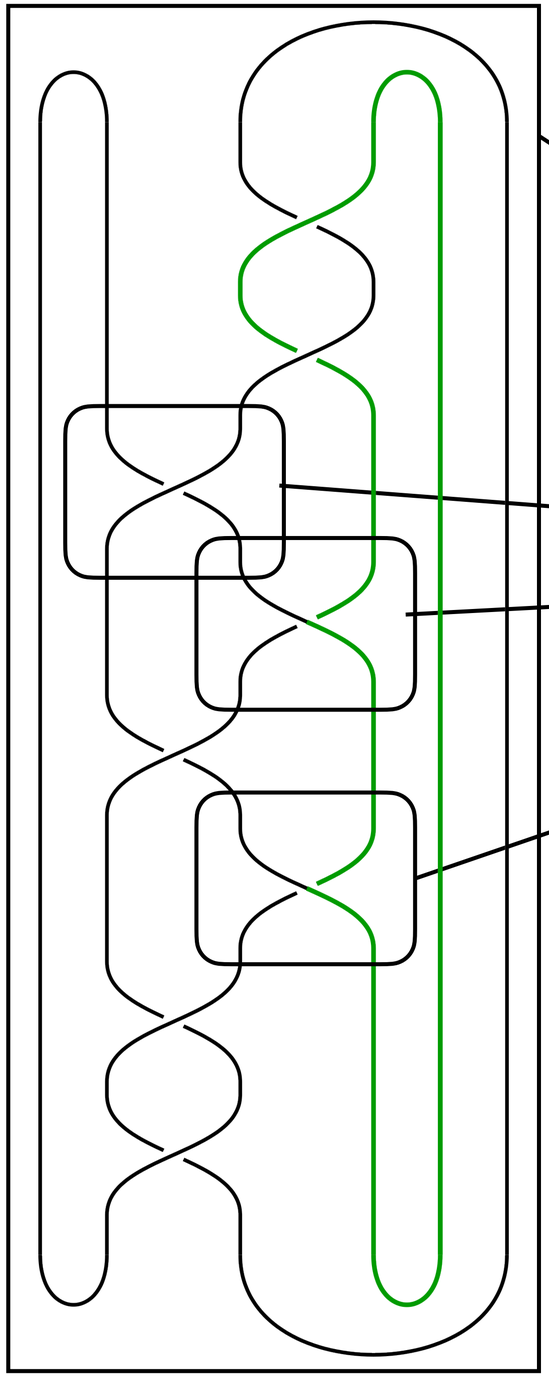
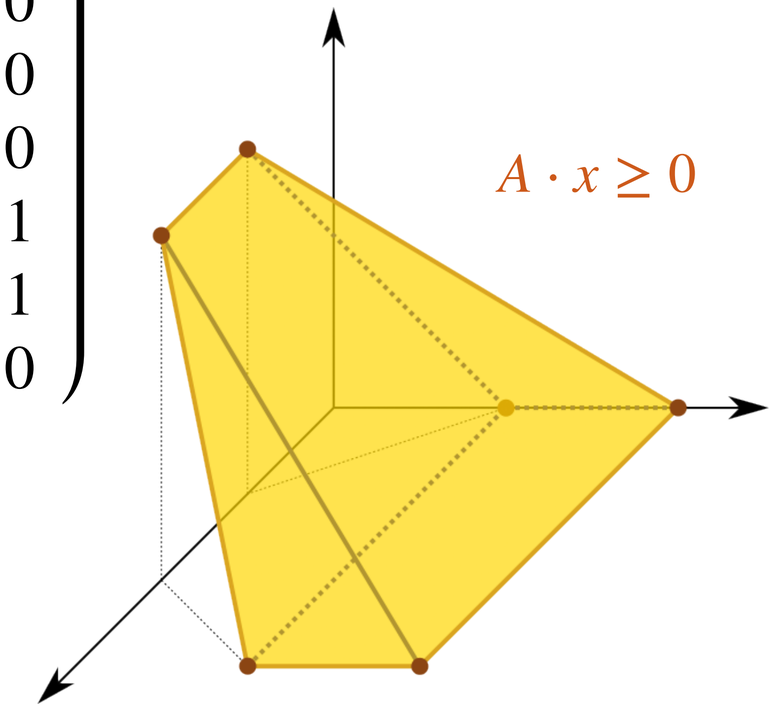

Integer Linear Programming (ILP)
Cycle inversion
Verification
braid space $\mathscr{B}$

Image: wikipedia.org.



$$A = \begin{pmatrix} 3.5 & -1 & -0 & -1 & -2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \end{pmatrix}$$



If $A \cdot x \geq 0$, the resulting state sum is a Laurent power series in q, x

Image: wikipedia.org.

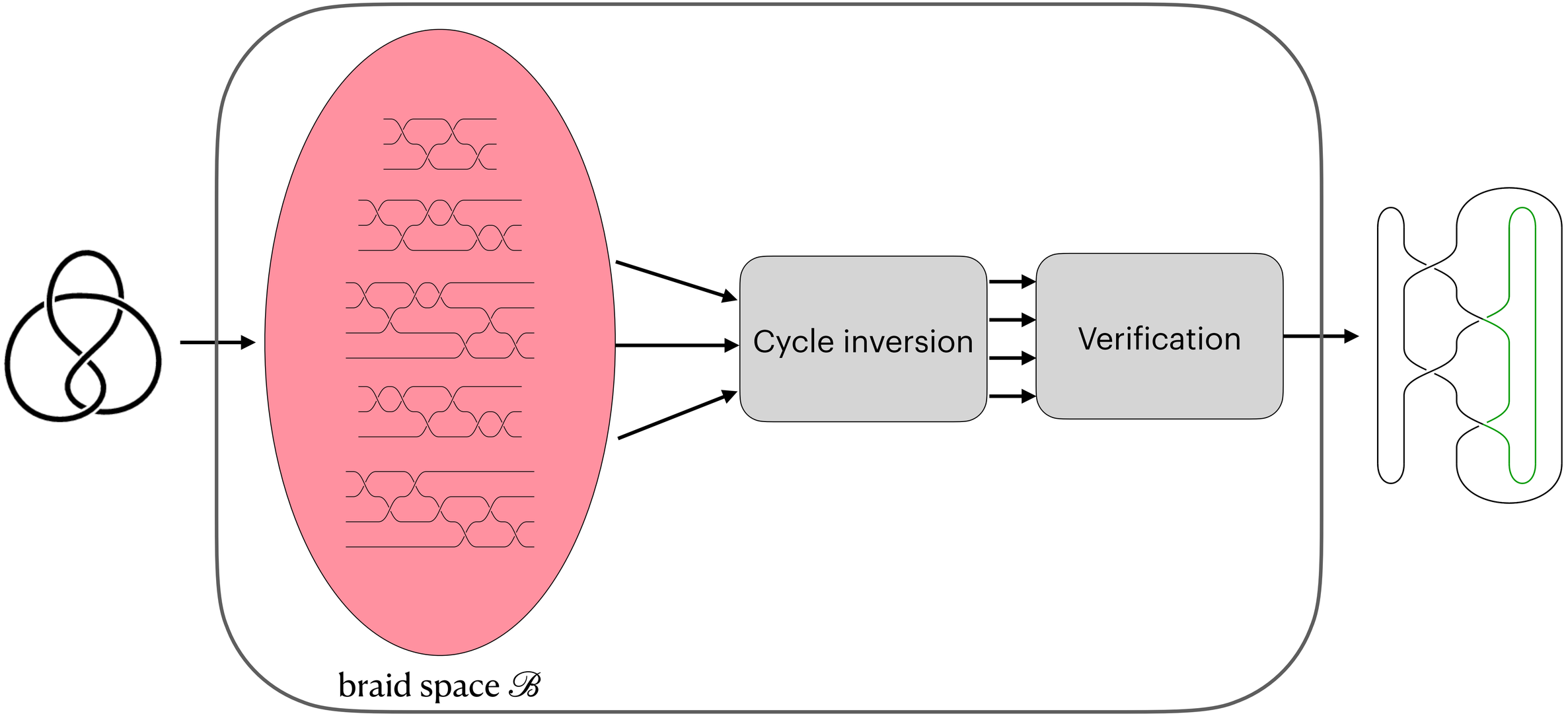
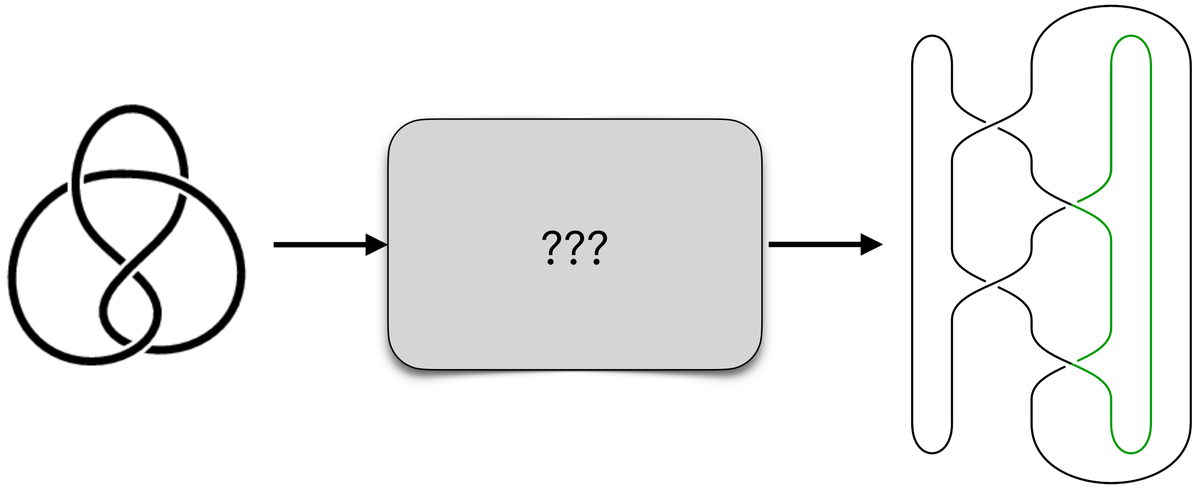


Image: wikipedia.org.

Can we *learn* this procedure?



The database

Result: 1,246 braids & choice of cycles

First large-scale computation of the F_K invariant

Gukov-Manolescu series (F_K)

This table contains the the first few terms of the Gukov-Manolescu series under the column fk_polynomial. The braid and inversion datum used to compute the inverted state sum are listed under braid and inversion_data.

1,246 records | Search... | Search | CODE | CITE | DOWNLOAD | PYTHON | MATHEMATICA | API

KNOTINFO_ID ?	BRAID ?	INVERSION_DATA ?	FK_POLYNOM
3_1	[1, 1, 1]	[-1, 1, -1, 1, -1, 1]	-q+q^2*x^2-
4_1	[1, -2, 1, -2]	[-1, 1, -1, 1, -1, 1, -1, 1]	-1-2*x-q^-
5_1	[1, 1, 1, 1, 1]	[1, 1, 1, 1, 1, 1, 1, 1, 1, 1]	-q^2*x+q^3
6_2	[1, 1, 1, -2, 1, -2]	[-1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1]	-q*x-2*q*x
6_3	[1, 1, -2, 1, -2, -2]	[1, 1, 1, 1, -1, 1, 1, 1, -1, 1, -1, 1]	-x-2*x^2+q



https://topology.fyi

First results

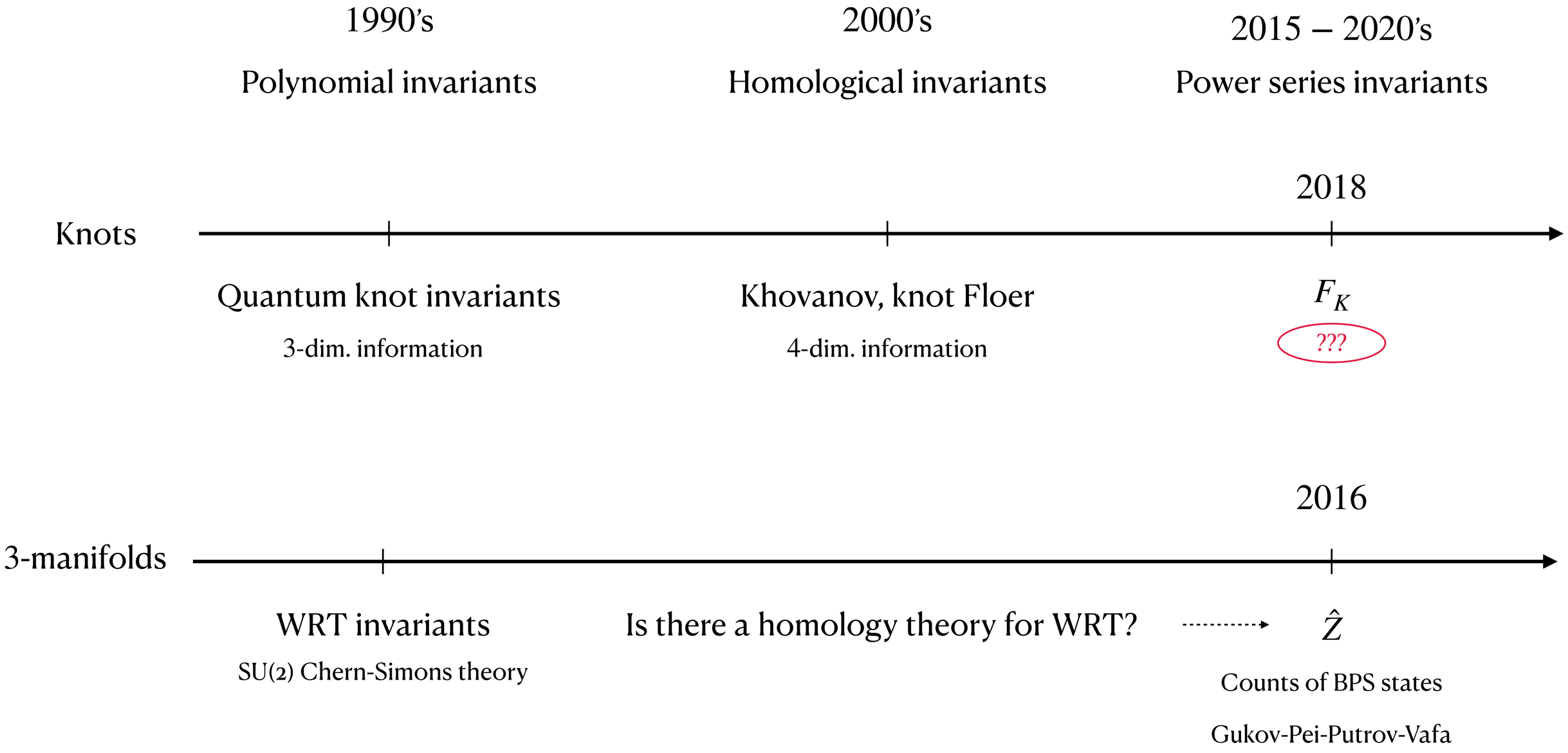

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Image: katlas.org

Theorem (Orland—Saunders-A'Court—S.—Svoboda): For a "nice" knot K,

$$F_K(x, q) = x^{-1/2}(q^{\ell_K} x^{d_K} + \text{higher order terms in } x)$$

Image: katlas.org

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If K is a homogeneous braid knot, or a **fibered knot of at most to 12-crossings**,

$$d_K = \max\{a \mid \widehat{HFK}(K,a) \neq 0\} = g_K$$

$$\ell_K = g_K - \min\{m \mid \widehat{HFK}_m(K, g_K) \neq 0\}$$

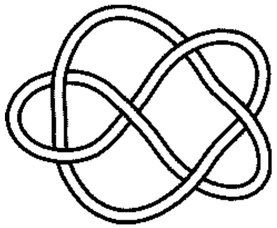
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$$\widehat{HFK}(6_2) = m\,a^2 + 3a + 3m^{-1} + 3m^{-2}a^{-1} + m^{-3}a^{-2}$$
$$F_{6_2}(x,q) = x^{-1/2}(q\,x^2 + 2q\,x^3 + (1 + 3q - q^2)\,x^4 + (2q^{-1} + 2 + 4q - 2q^2)\,x^5 + \cdots)$$

Corollary (Orland—Saunders-A'Court—S.—Svoboda): For a "nice" knot K,

$$\ell_K = \sum_{i=0}^{2g_K-1} \frac{1}{i!} \frac{d^i \Delta_K^3}{dy^i}\bigg|_{y=0} \frac{dh^K_{2g_K-i}}{dq}\bigg|_{q=1}$$

Habiro coefficients
(Change of basis of Jones polynomials)

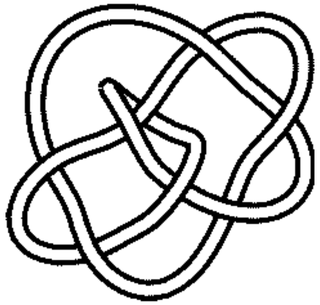
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(Change of basis of Jones polynomials)
Habiro coefficients

Corollary (Orland—Saunders-A'Court—S.—Svoboda): If K is a homogeneous braid knot, or a **fibered knot of at most to 12-crossings**,

$$\min\{m \mid \widehat{HFK}_m(K, g_K) \neq 0\} = g_K - \sum_{i=0}^{2g_K-1} \frac{1}{i!} \frac{d^i \Delta_K^3}{dy^i}\bigg|_{y=0} \frac{dh^K_{2g_K-i}}{dq}\bigg|_{q=1}$$

Homological grading



$$\begin{aligned} 4 &= 3 - (2030 - 8505 + 14076 - 11508 + 4662 - 756) \\ &= 3 - (-1) \end{aligned}$$

Let $F_K = \sum_{n=0}^{\infty} f_n(q)\, x^n$ and $\delta_n(K) := \min \deg_q f_n(q)$

Conjecture (Orland—Saunders-A'Court—S.—Svoboda): For a "nice" knot K,

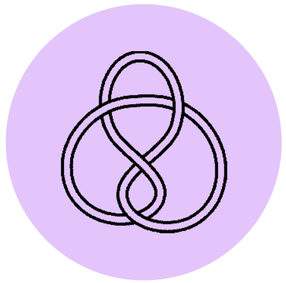
$$\delta_n(K) = An^2 + B(n)\, n + C(n)$$

where A is the inverse of a boundary slope in $S^3 \backslash K$.

Determines whether the surgery formula works!

Image: katlas.org, vecteezy.com

Knot complement



$S^3 \backslash K$

$$F_K = x^{1/2} \sum_{n=0}^{\infty} f_n(q)\, x^n$$

surgery formula

$$\oint_{|x|=1} \frac{dx}{2\pi i x} \cdots$$



Dehn surgery

3-manifold M_3

$$\hat{Z}(M_3) = q^{\Delta_b} \left(\sum_{n=0}^{\infty} a_n\, q^n \right)$$



Let $F_K = \sum_{n=0}^{\infty} f_n(q)\, x^n$ and $\delta_n(K) := \min \deg_q f_n(q)$

Conjecture (Orland—Saunders-A'Court—S.—Svoboda): For a "nice" knot K,

$$\delta_n(K) = An^2 + B(n)\, n + C(n)$$

where A is the inverse of a boundary slope in $S^3 \backslash K$.

Determines whether the surgery formula works!

Understanding the choice will help us solve the 

Image: katlas.org

Non-perturbative complex Chern-Simons theory

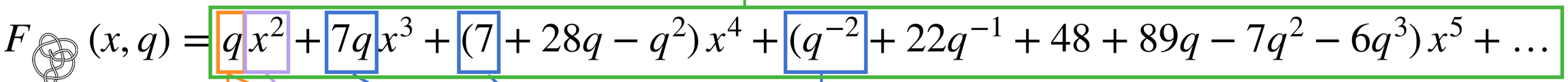
$q \to w_p$: colored Jones, ADO, Alexander...

$$F_{\text{}}(x,q) = q\,x^2 + 7q\,x^3 + (7 + 28q - q^2)\,x^4 + (q^{-2} + 22q^{-1} + 48 + 89q - 7q^2 - 6q^3)\,x^5 + \dots$$

Image: katlas.org

Non-perturbative complex Chern-Simons theory

$q \to w_p$: colored Jones, ADO, Alexander...



Boundary slopes of essential surfaces

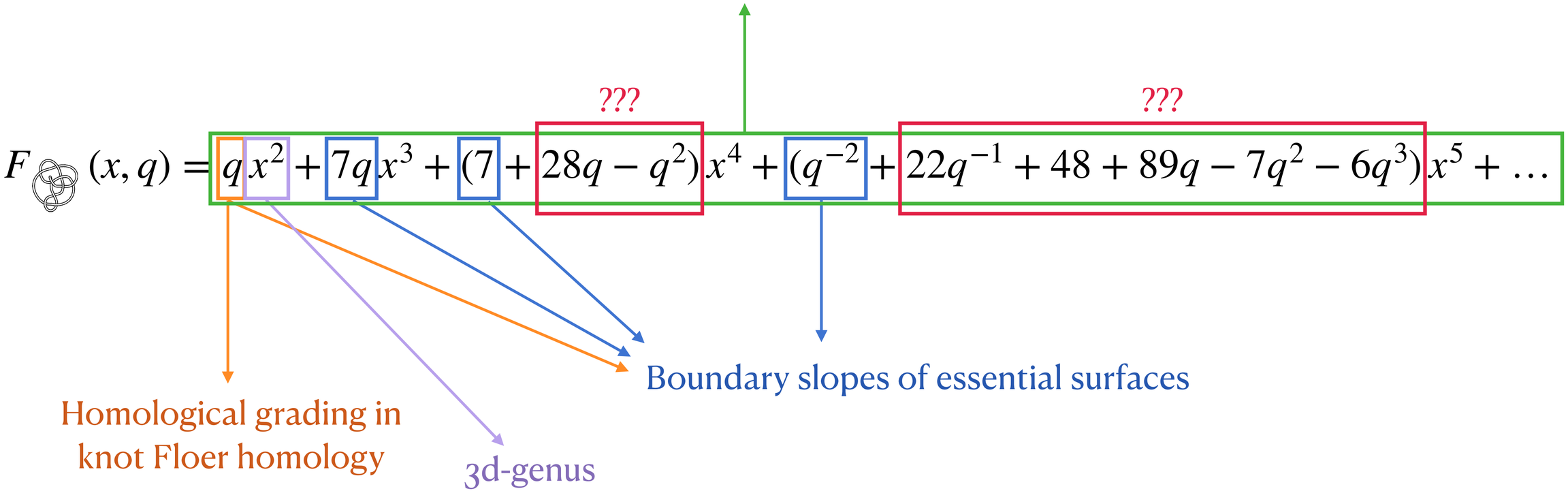
Homological grading in knot Floer homology

3d-genus

Image: katlas.org

Non-perturbative complex Chern-Simons theory

$q \to w_p$: colored Jones, ADO, Alexander...





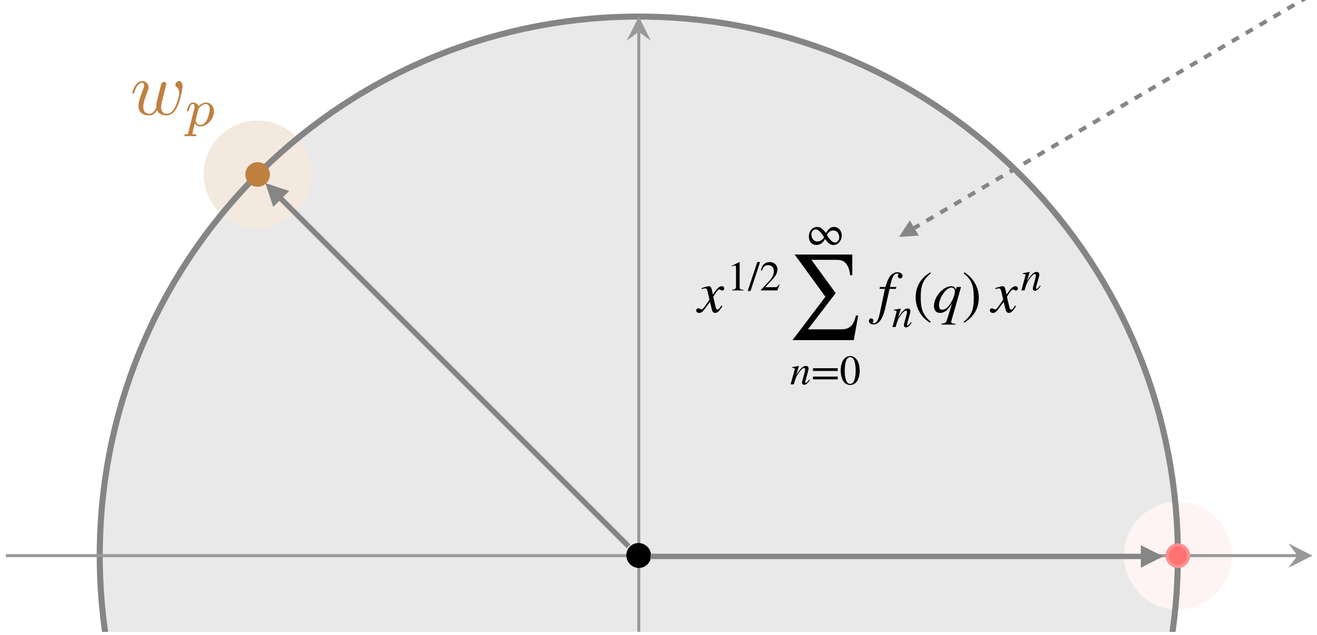
https://topology.fyi

OSSS., 2025

Gukov-Manolescu, 2018: resurgent analysis at $q = 1$

S. Park, 2020–21: quantum groups at generic q

$$\frac{ADO'_p}{\Delta_K(x^p)} + \sum_{j=1}^{\infty} \hbar_p^j \frac{P^{(j)}_{K,p}(x)}{\Delta_K^{2j+1}(x^p)}$$



$$\frac{x^{1/2} - x^{-1/2}}{\Delta_K(x)} + \sum_{j=1}^{\infty} \hbar^j \frac{P^{(j)}_K(x)}{\Delta_K^{2j+1}(x)} \qquad q = 1 + \hbar$$

Alexander polynomial: $\Delta\left(\times\right) - \Delta\left(\times\right) = \left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)\Delta\left(\times\right) \qquad \Delta\left(\bigcirc\right) = 1$